Journal of Mathematical Behavior 43 (2016) 111–133

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

The unit circle approach to the construction of the sine and cosine functions and their inverses: An application of APOS theory

Rafael Martínez-Planell*, Angel Cruz Delgado

University of Puerto Rico at Mayagüez, Puerto Rico

ARTICLE INFO

Article history:
Received 13 November 2015
Received in revised form 16 June 2016
Accepted 22 June 2016
Available online 18 July 2016

Keywords:
APOS
Trigonometry
Unit circle
Inverse trigonometric functions
Genetic decomposition

ABSTRACT

We use Action-Process-Object-Schema (APOS) Theory to analyze the mental constructions made by students in developing a unit circle approach to the sine, cosine, and their corresponding inverse trigonometric functions. Student understanding of the inverse trigonometric functions has not received much attention in the mathematics education research literature. We conjectured a small number of mental constructions, (genetic decomposition) which seem to play a key role in student understanding of these functions. To test and refine the conjecture we held semi-structured interviews with eleven students who had just completed a traditional college trigonometry course. A detailed analysis of the interviews shows that the conjecture is useful in describing student behavior in problem solving situations. Results suggest that students having a process conception of the conjectured mental constructions can perform better in problem solving activities. We report on some observed student mental constructions which were unexpected and can help improve our genetic decomposition.

1. Introduction

Periodic or nearly periodic phenomena abound in nature and in the design of man-made machinery. A simple such case, a particle moving on a circle at a constant rate, will lead to the establishment of relations between the particle's position and the arc length it has traveled from a given starting point, or its position angle if you will. This kind of behavior is modeled with sinusoidal functions, a basic and important component of mathematics. Indeed, sinusoidal functions appear, for example, as generators of solutions to widely applied differential equation models, and as the basic elements of Fourier series. Students of mathematics, physics and engineering are bound to meet periodic phenomena and hence will benefit from understanding and being able to apply the properties of this basic family of functions.

We choose to center attention on the simple nature of a circle and the description of the changing position of a particle traversing around the circle as this is a most intuitive and natural idea that can serve as a basis and organizing idea from which to derive other knowledge about periodic behavior. Hence, our interest here is to look at student understanding of unit circle trigonometry. This choice brings forth the subtle interplay between geometric and analytical thinking involved in the mental construction of the sine and cosine functions which is a potential trouble spot for some students. Further, this choice will also allow us to discuss what is normally regarded as a difficult idea for students: to invert the sinusoidal

* Corresponding author.
E-mail addresses: rafael.martinez13@upr.edu (R. Martínez-Planell), angel.cruz14@upr.edu (A. Cruz Delgado).

http://dx.doi.org/10.1016/j.jmathb.2016.06.002
0732-3123/

functions by the reversal of the unit circle process that leads to their definition. While students may be able to do many of the trigonometry related tasks commonly requested of them by relying on technology and memorization, unit circle trigonometry is an organizing principle needed for deeper understanding.[1]

Our research questions are:

1. What mental constructions can be conjectured that students could do in order to have a process conception (as understood in APOS Theory) of sine, cosine, and their corresponding inverse functions?
2. Which of the conjectured mental constructions can be inferred from students work when they are involved in problem solving activities that involve the sine and cosine functions and their inverses?
3. Which of the conjectured mental constructions seem to be lacking in the work of students that have difficulties or appear to be present in the work of those that succeed while facing problem solving activities involving the sine and cosine functions and their inverses?

2. Background

Numerous studies have documented that students frequently show limited understanding of basic ideas of trigonometry (Bagni, 1997; Pritchard & Simpson, 1999; Weber, 2005; Brown, 2005; Moore, 2010).

Bagni (1997) noted that more than 80% percent of the 67 Italian high school students in his study could provide a complete or partial solution to easy trigonometric equations such as, find all real values x such that $\sin x = -1/2$ or $\cos x = 1/2$, by remembering and mentally reversing a memorized table of the values of trigonometric functions of integer multiples of $\pi/6$, and $\pi/4$. He reported, that more than half of the students tested produced wrong answers or no answer to questions such as find all real values x such that $\sin x = 1/3$, $\sin x = \pi/3$, or $\cos x = -\sqrt{3}/3$. For Bagni, the erroneous responses to these more difficult questions are examples of the effect of the didactical contract.[2] His paper provides an interesting illustration of how the particular choices and interpretations of both explicit and implicit learning trajectories, learning goals and expected learning outcomes, play a role on what students focus their attention as they learn. What is perceived as important by students, namely, the expectation that an equation must always have a solution or the values of the trigonometric functions at special values, may as well prompt an automatic response devoid of meaning. In his paper Bagni also writes the seemingly innocuous statement: "From that, any student would think: what is the meaning of this exercise? Does it exist, can it [sic] exist a value of x such that $\sin x = 1/3$?" While the trained mathematician may see this question as natural, Bagni's work implicitly raised a more fundamental question: which thought mechanisms come into play in students minds while they perform this task? By itself, the equation is a statement that encapsulates the application of multiple and interconnected processes in which the meaning of mathematical objects from the symbolic, geometric and numeric realms play an important role. To pose and to fully answer the question a novice must first build a non-trivial set of mental connections between the relevant facts, procedures and ideas related to the problem. While achieving a certain level of proficiency as to being able to interact with a community of people that do and make sense of mathematics is an important objective by itself, we view the need to describe and understand well the cognitive roadmap and productive struggle necessary to achieve the goal of solving the equation as an indispensable tool for instruction.

Weber (2005) noted that while algebraic functions deal with arithmetic operations and procedures, the reasoning behind trigonometric functions arises from the geometric realm in which construction and measurement are important implicit ideas. Weber's work views the trigonometric functions as operations applied to angle measurements and is mainly concerned with examining whether students can explain or derive some of their properties. He conjectured that students need to be able to imagine the process (from angle to circle to value) which gives rise to the unit circle definition of the trigonometric functions in order to be able to have an understanding of the sine and cosine functions that goes beyond the repetition of memorized facts and procedures (the understanding shown by students in Bagni's 1997 study). Weber also observed two limitations in how the college students exposed to traditional instruction that participated in his study understood trigonometric functions: first, students appeared to perceive these functions as external operations prompted by step by step prescriptions, a set of cues or labeled diagrams that dictate the algorithm to follow (an action conception as defined in APOS; see also Breidenbach, Dubinsky, Hawks, & Nichols, 1992), and second, "What these students seemed to lack was the ability or inclination to mentally or physically construct geometric objects to help them deal with trigonometric situations" (Weber, 2005). Weber traced and applied experimentally a learning trajectory that included specific attention to the mathematical process of constructing the trigonometric functions. It included some activities in which students find the sine and cosine values of an angle using a protractor and a unit circle. A high proportion of the students under this experimental instruction were able to estimate the values of the trigonometric functions of non-standard angles, articulate the process of finding the

[1] Throughout this article, when we refer to "a deeper understanding" or any such phrase that implies an understanding that goes beyond the mere application of memorized facts or procedures, we mean a process conception as defined in APOS Theory (see ahead).

[2] "The teaching of mathematics is a social project of putting at the disposal of all the members of a society the means of participating in a common mathematical culture and benefiting from it. To each precise notion to be taught, the partners in teaching (i.e. the teachers, the learners, and the other parties mentioned above) associates expectations, obligations that each undertakes and benefits from, and the means by which they envisage (mutually or separately) satisfying these expectations and obligations as well as the consequences of not satisfying them. A *didactical contract* is, in the broad sense, an interpretation of the set of these expectations and obligations, be they compatible, explicit, and agreed to or not" (see Brousseau and Warfield, 2014).