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An examination of college mathematics majors' understandings of their own written definitions

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ABSTRACT

This qualitative study of ten undergraduate mathematics majors examined students' abilities to write definitions and found that students at the advanced level of undergraduate mathematical study have difficulty creating definitions that conform to their concept images or to accepted definitions of basic concepts. This is due in part to (1) failure to consider key examples when writing definitions, (2) weak concept images for the concept in question, and (3) vague concept images for related concepts. The results of this study have implications for secondary-level and college-level mathematics instruction.

1. Introduction and framework

Mathematical proofs have long been known to be difficult for students of mathematics. Many misconceptions have been identified as well as many areas of difficulty. Moore (1994) listed seven possible reasons for students' inability to write proofs and most had something to do with either mathematical language or definitions. Vinner (1977) suggests that students do not know how to use definitions in proofs. This may be because students must first fully *understand* definitions before they can use them. In this study, we wanted to investigate students' use of definitions that they understood. We did this by first asking them to provide for us their own definitions of familiar concepts, and then asking them to use those definitions to make determinations. What resulted was a cycle of refinement of definitions in which participants attempted to bring their definitions into closer agreement with their concept images.

1.1. The nature of mathematical definitions

Definitions play a central role in mathematics. Mathematicians and students of mathematics use definitions routinely but seldom think about the nature of mathematical definition (Wilson, 1990). One use for a mathematical definition is really as a mathematical short-hand or abbreviation. We define a function $f : D \to \mathbb{R}$ to be continuous at some c in D provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$. From that moment on, we do not have to refer to such functions as "functions with the property that for every $\varepsilon > 0$ there is a $\delta > 0$ such that..." We just call such functions "continuous." The process of defining in mathematics is merely the creating of names to go with concepts—typically useful or interesting concepts.

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Definitions in mathematics are somewhat different than they are in natural language in that they are *stipulative* rather than *extracted*. "Extracted definitions report usage, while stipulative definitions create usage" (Edwards & Ward, 2004, p. 412). New words emerge all the time by common usage and extracted definitions describe that usage. Extracted definitions *describe* how a word is used and what is meant by those who use it. By contrast, mathematical definitions are *stipulative*. They are created on the advice of experts and *prescribe* how a word shall be used and what it shall mean.

Beyond being created on the advice of experts, mathematical definitions need no further justification. They can neither be proven nor disproven, only accepted or rejected. Consider the following commonly accepted definition: A quadrilateral is called a *trapezoid* provided it has at least one pair of parallel sides. Under this definition, a square would be a trapezoid. If we are satisfied that a square should be considered to be a trapezoid, then we can *accept* this definition. If not, we can *reject* it in favor of one that disqualifies squares from consideration.

1.2. Students' use of mathematical definitions

To have at least a partial understanding of a concept means to have an image of it. The term *concept image* is used "to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). Thus, being able to recite a definition of a concept from rote memory does not guarantee a robust understanding of the concept (Vinner, 1991). Tall and Vinner (1981) report that students' concept images may be quite different from the definitions of those concepts. Thus, when students write proofs from faulty or incomplete concept images, even a correct line of reasoning can lead one astray.

Moore (1994) identified several difficulties students have in writing mathematical proofs including: students often (1) do not know the definitions, (2) have little intuitive understanding of the concepts in question, (3) have concept images that are inadequate for doing proofs, (4) do not generate and use their own examples, (5) do not know how to use definitions to obtain the overall structure of a proofs, (6) neither use nor understand mathematical language, and (7) do not know how to begin a proof. All but the last of these has something to do either with definitions, or with mathematical language. This list is by no means exhaustive. Perhaps most notably absent from the above list is students' difficulties with the underlying logic of proofs. Harel and Sowder (1998) observed that while students in their study attempted to prove theorems by example (often a single example), they seldom attempted to refute them by counterexample and were generally unconvinced by disproofs of this method. More recently, in working with 23 students of advanced undergraduate mathematics, Wheeler and Champion (2013) identified several errors and misconceptions that students displayed when writing proofs. Among the difficulties concerning the underlying logic of proofs were (1) beginning with the conclusion, (2) interfering knowledge (modeling an *onto* proof after a *one-to-one* proof), and (3) proof by example.

Edwards and Ward (2004) reported that students do not use mathematical definitions in the same ways that mathematicians do. Even students who can state and explain definitions often reason from their concept images. When concept images and concept definitions conflicted for two of their participants, they apparently believed that the definition had not been correctly extracted and argued incorrectly from their concept images instead. Dickerson and Pitman (2012) reported that reasoning entirely from incomplete concept images was common among college mathematics majors even when the definitions were given, and rejecting the given definitions in favor of vague concept images was common. In contrast, while mathematicians also reason from their concept images, the reasoning given in formal proofs is based on definitions. Moreover, if confronted with difficulties while reasoning from their concept image, mathematicians defer to the definition.

1.3. Methods

The questions that guided our study are: (1) What difficulties do students encounter when attempting to write definitions to match their concept images? and (2) How do students use definitions of their own creation? Ten college mathematics majors participated in this study. Each was currently a student in an upper division course in college geometry or algebraic structures. Participants filled out a paper survey and each was interviewed twice by the authors. The data for this study comes from the survey and the interviews.

In the survey, participants were asked to define the terms *even*, *prime*, *square*, *parallel lines*, *tangent line*, and *trapezoid*, as carefully as they could. We felt that college mathematics majors would have fairly robust concept images of these middle school and high school level concepts and would be more successful at writing definitions for them than they would be for concepts such as *ring* or *uniform continuity*. It is important to note that some of these terms are defined differently from author to author. For example, *prime* is defined by some in such a way that allows for negative numbers to be prime. Our intent was not to determine whether our participants' concept images would allow primes to be negative, squares to be trapezoids, or zero to be even; but whether they could write definitions consistent with their concept images and then use their written definitions to make determinations of examples and non-examples. After reading each survey, the two authors met to create follow-up questions to ask each participant as the basis for the first of two semi-structured interviews.

The interviews were semi-structured, audio-recorded, and transcribed. The interview was task-based (Goldin, 2000) in which we asked participants to re-read the definitions they provided on the survey and then to respond to approximately three or four follow-up questions for up to five of their definitions to probe their understanding. Our follow-up questions were meant to determine whether the written definition really captured the participant's concept image, and if not to provide an opportunity for the participant to amend their definition to better describe their concept image. For example,