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Co-construction of fractions schemes and units coordinating structures

Steven Boyce[a,*], Anderson Norton[b]

[a] Portland State University, Portland, OR, USA
[b] Virginia Tech, Blacksburg, VA, USA

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ABSTRACT

A growing body of research implicates students' ability to coordinate multiple levels of numerical units as an important aspect of their mathematical development. In this paper, we consider relationships between the ways students coordinate units with whole numbers (their multiplicative concepts) and the ways students coordinate units with fractions (their fractions schemes). Interviews with 50 sixth-grade students suggest commonality in the number of levels of units-within-units structures that students construct for the two contexts, consistent with Steffe's (2001. *The Journal, of Mathematical Behavior*, *20*(3), 267) reorganization hypothesis. The results suggest that fractions may be a fertile domain for elementary and middle grades students to continue to develop whole number understandings.

1. Introduction

Steffe's (2001) reorganization hypothesis posits that students' ways of thinking about fractions – their fractional schemes – can result from their reorganization of the psychological operations they develop as they construct schemes for working with composite units. However, many students have yet to construct fractions understandings beyond part-whole comparisons by the time they reach middle school. As part of our ongoing efforts to develop ways to support such students, we assessed 50 sixth-grade students' whole number multiplicative concepts and fractions schemes, using items adapted from Hackenberg and Lee (2015) and Norton and Wilkins (2010). Our results indicate that students entering the middle grades are still developing structures for coordinating multiple levels of units, but students generally assimilate the same number of levels of units in whole number contexts as they do in fractions contexts. Analysis of the outlying cases suggests potential for "reverse" or "parallel" reorganizations for some students—fractions may be a fertile domain for middle-grades students to continue to develop whole-number multiplicative concepts.

2. Literature review

Despite decades of research efforts, students' difficulty understanding fractions persists (Mullis, Martin, Foy, & Arora, 2012). Although it is critical for students to develop ways of thinking about fractions beyond initial part-whole comparisons (Behr, Harel, Post, & Lesh, 1992), students commonly lack experiences from which more powerful meanings for fractions

* Corresponding author. Tel.: +1 5037254833; fax: +1 5037253661.

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might develop, as do many of their teachers (Izsák, Jacobson, de Arajuo, & Orrill, 2012). These other meanings are typically described in terms of measure, operator, ratio, and quotient sub-constructs of fractions (Kieren, 1979).

A robust understanding of fractions includes developing each of the fraction sub-constructs, but developing an understanding of a fraction as a measure, such as a length on a number line, has been identified as particularly important for students' future mathematics achievement (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010; Siegler, Thompson, & Schneider, 2011). There is growing evidence that students' ability to think of fractions as measurable quantities affords early algebraic reasoning (Hackenberg, 2010; Hackenberg & Lee, 2015; Norton & Wilkins, 2012; Olive & Çağlayan, 2008).

When Lamon (2007) studied the influences of varying the focus of introductory fractions instruction with second-graders, she found that emphasizing the measurement construct was the most effective for helping her students to develop the other interpretations of fractions. But while Lamon's (2007) study suggested benefits to introducing fractions as measures, she found that the most important characteristic affecting individuals' ability to learn fractions concepts was not the sub-construct she emphasized in instruction: "[A] more significant factor in overall success was the development of the central multiplicative structures" (Lamon, 2007, p. 660).

Mathematics education researchers from diverse theoretical frameworks have modeled structural relations between addition, multiplication, exponentiation, fractions, and rational numbers (*e.g.*, Behr et al., 1992; Confrey & Smith, 1995; Lamon, 2007; Hackenberg, 2007; Izsák, Orrill, Cohen, & Brown, 2010; Norton & Wilkins, 2012; Piaget, 1970b; 2001; Streefland, 1993; Thompson & Saldanha, 2003; Vergnaud, 1994). The notion of structure serves dual purposes in modeling the development of rational number concepts. There is consideration of the mathematical structure that might *result* from learning (*e.g.*, understanding of the rational number system as a field), and there is consideration of the psychological structures learners construct *as they* are learning (*e.g.*, the structure of children's counting schemes). In this paper, we focus on the latter to analyze relationships between sixth-grade students' multiplicative concepts and their fractions schemes.

3. Theoretical framework

We adapt a scheme-theoretic perspective (Von Glasersfeld, 1995) to include an interpretation of Piagetian structure (1970b to consider relationships between students' units coordination with fractions and their units coordination with whole number multiplication.

3.1. Von Glasersfeld's scheme theory

A scheme is an individual's established way of experiencing and operating in service of a goal. A scheme consists of three sequential components: a recognition template, mental activity, and an expected result (Von Glasersfeld, 1995). When the result of goal-directed activity is unexpected, resolving this *perturbation* is often an impetus for establishing a new scheme.

The recognition template of a scheme is its assimilatory structure, which is obtained from the sensorimotor pattern awareness or mental imagery associated with an activity. Assimilation involves modification of perceptual input so that it fits into an individual's existing conceptual structures (Von Glasersfeld, 1995). In the beginning, *participatory* stage of a scheme, the recognition template is closely tied to particular contexts and recent activities (Tzur, 2007).

An *operation* is an interiorized mental action. Operations are abstractions of actions that "can be carried out through representation and not through actually being acted out" (Piaget, 1970a, p. 14). This abstraction is what allows an individual to coordinate operations with other operations. In contrast, the results of mental activity that are someone has merely internalized must still be carried out, at least in his imagination.

Reflective abstraction is the process by which internalized actions previously tied to sensorimotor activity become interiorized (Von Glasersfeld, 1995), by which the recognition template becomes less dependent on specific contexts and schemes become more *anticipatory* (Tzur, 2007). When an entire tri-partite scheme structure is interiorized, a scheme is called a *concept* (Hackenberg, 2007). The recognition template and the expected result of a concept are perceived simultaneously, which is what allows concepts to become parts of the assimilatory structure of other schemes.

3.2. Piagetian structures

Piaget (1970b) defines a *structure* as comprising "three ideas: the idea of wholeness, the idea of transformation, and the idea of self-regulation" (p. 5). *Wholeness* refers to the idea that, although structures have individual elements, the relationships between elements are defined by global rules that apply to a totality. *Transformation* refers to the idea that structures are not static forms; they represent a potential for "intelligible change that always preserves invariance in certain respects" (p. 20). *Self-regulation* refers to the idea that the "transformations inherent in a structure never lead beyond the system, but always engender elements that belong to it and preserve its laws" (p. 14). The versatility of the construct permits flexibility in analyzing both recurrent and nested structural relationships.

A scheme is an example of a structure, for organizing a sequence of operations in service of a goal. "Piaget (1970a, 1972) considered a scheme to be a general structure of a set of actions that conserves itself by repetition, consolidates itself by exercise, and applies itself to situations that vary because of contextual modification" (Hunting, Davis, & Pearn, 1996, p. 356). Whereas a person engages in acts of self-regulation while carrying out the activity of a scheme, transformations that are