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Shared communication in building mathematical ideas: A longitudinal study

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ABSTRACT

Students make sense of mathematical ideas using a variety of representations including physical models, pictures, diagrams, spoken words, and mathematical symbols. As students' understanding of mathematical ideas becomes more general and abstract, there is a need to express these ideas using mathematical notation. This paper describes students' movement from model building and personal notations to elegant use of mathematical symbols that show their understanding of advanced counting ideas. Specifically, this paper shows how earlier ideas from investigations of specific combinatorics problems (questions about making pizzas with different toppings and using cubes to build towers) are retrieved and built upon using the formal mathematical register to explain the meaning of Pascal's Identity, the addition rule of Pascal's Triangle. This analysis also shows the power of shared communication in mathematical problem solving.

1. Introduction

The mathematical register can be described as the language of mathematics that is used to express mathematical ideas (Halliday, 1978). In this work I trace, over an eight-year period, the developing use of the mathematical register of a group of students who were participants in a long-term study of the development of mathematical ideas and ways of reasoning (Maher, 2005, 2010; Muter, 1999). I focus on student work on problem tasks in counting and combinatorics beginning in the elementary grades and continuing through middle school and high school. By high school, the students demonstrated understanding of the structure of the solutions to several problems in combinatorics. They related what they understood to representations of Pascal's Triangle. Finding their personal notations inadequate for the challenge of generalizing their findings, the students made use of standard mathematical notation to represent Pascal's Identity.

1.1. Objectives

There are two objectives in this paper: (1) to show how the students' ways of communicating about and representing their thinking evolved from idiosyncratic and personal to general and shared; (2) to trace how their focus moved from surface features of attributes of the tasks to the mathematical structure shared by both tasks. In particular, this paper describes how the students came to represent Pascal's Identity using standard notation for combinations.

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In addition to tracing the evolution of personal notation to formal representation, the analysis shows how the students communicated their understandings to each other and to other observers using standard mathematical terminology. In the students' journey from use of personal notations to use of the mathematical register, they combined their understanding of the specific combinatorics problems with their knowledge of standard mathematical notation in order to express Pascal's Identity; in that way, they displayed their understanding of the structure of the general solution.

1.2. Overview

The paper is organized into seven sections:

- The introduction provides the setting for the research.
- The second section provides the theoretical framework guiding the analyses; this includes shared communication among learners and how their personal notations developed over time as they built mastery of the mathematical register.
- The methodology section describes data collection, analyses, and procedures; this section also provides a detailed description of the mathematical problems that form the focus of this study (the towers and pizza problems).
- The results sections traces how the students dealt with the towers and pizza problems and how, over time, they discovered the isomorphic relationship between the pizza and towers problems and the isomorphic relationship between each of these problems and other mathematical entities (e.g. the binomial expansion).
- The discussion section summarizes the results.
- The sixth section discusses pedagogical implications.
- The concluding section discusses implications for practice.

2. Theoretical framework

The framework that guides this analysis relates to communication among learners of mathematics and representations used by learners of mathematics. The importance of communication among learners has been documented extensively (e.g. Maher, 2005; Morgan, 2006; Sfard, 2000; Staats & Batteen, 2010). Research shows that students can and should use personal notations when exploring new mathematical ideas (Maher, Sran, & Yankelewitz, 2010; Martino & Maher, 1999). However, students are expected eventually to master the standard mathematical register in order to generalize and communicate their mathematical ideas (National Council of Teachers of Mathematics, 2000; O'Halloran, 2005; Skemp, 1987).

2.1. Representations and the mathematical register

As noted by Cuoco, the idea of representations can be difficult to define precisely:

> All of us have an intuitive idea of what it means to represent a situation; we do it all the time when we teach or do mathematics. . . . But what do we mean, precisely, by "representation," and what does it mean to represent something? These turn out to be hard philosophical questions that get at the very nature of mathematical thinking. (Cuoco & Curcio, 2001, p. *x*)

In fact, in many instances, the meaning of "representation" is left undefined. For example, the Common Core State Standards for mathematics (CCSS, 2010) discuss the use of representations and require that students use and make sense of various representations but the term "representation" is never defined. In this paper, I focus on external representations (those that are observable) and I define them by way of example. Pictures, models, words, and symbols are representations that will be discussed in this paper. In addition, students' personal representations – models, words, pictures, and symbols developed by students – will be contrasted with the type of representation called the standard mathematical register.

The mathematical register can be considered the language of mathematics used to express mathematical ideas. Since Halliday's early work (1978), other researchers have expanded on this definition. For example, Pimm (1987) states that the language of mathematics is characterized by its use of specialized words and phrases and of the use of everyday words with specialized meanings. In addition, O'Halloran (2005) lists three semiotic components of the mathematical register as "language, visual images and mathematical symbolism" (p. 11). However, there is no clear distinction between the mathematical register and regular language that is not part of mathematical discourse (Barwell, 2013; Moschkovich, 2008). Further, Barwell notes, "Everyday language does not disappear in mathematics; it is used in new, more mathematical ways" (p. 221). For purposes of this analysis, I follow O'Halloran, considering the mathematical register to be composed of images, mathematical symbols, and spoken or written language. I consider use of language to include use of specialized mathematical terms, use of everyday words with specialized mathematical meanings, and, following Barwell, ordinary language used in mathematical ways.

Proficiency in the mathematical register aligns with the CCSS goal that students should "attend to precision" (2010, p. 7) in their communication about mathematics. Furthermore, proficiency with the mathematical register enables students to achieve the CCSS requirement that high school students be able to "make explicit use of definitions" (p. 7). However, Barwell (2013) observes that students do not progress linearly from everyday language to formal use of the mathematical register; moreover, there is a place in mathematical discourse for everyday language. Students should not just learn to use