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Making sense of eigenvalue–eigenvector relationships: Math majors' linear algebra – Geometry connections in a dynamic environment

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ABSTRACT

The present qualitative case study on mathematics majors' visualization of eigenvector–eigenvalue concepts in a dynamic environment highlights the significance of student-generated representations in a theoretical framework drawn from Sierpinska's (2000) modes of thinking in linear algebra. Such an approach seemed to provide the research participants with mathematical freedom, which resulted in an awareness of the multiple ways that eigenvalue–eigenvector relationships could be visualized in a manner that widened students' repertoire of meta-representational competences (diSessa, 2004) in coordination with their preferred modes of visualization. Students' expression of visual fluency in the course of making sense of the eigenvalue problem $Au = \lambda u$ associated with a variety of matrices occurred in different, yet not necessarily hierarchical modes of visualizations that differed from matrix to matrix: (i) synthetic/analytic mode manifested in the process of detecting eigenvectors when the sought eigenvector and the *matrix-applied* product vector were aligned in the same/opposite directions; (ii) analytic arithmetic mode manifested in the case of singular matrices (in the determination of the zero eigenvalue) and invertible matrices with nonreal eigenvalues; (iii) analytic structural mode, though rarely occurred, manifested in making sense of the trajectory (circle, ellipse, line segment) of the *matrix-applied* product vector and relating trajectory behavior to matrix type. While the connection between the thinking modes (Sierpinska, 2000) and the concreteness–necessity–generalizability triad (Harel, 2000) was not sharp, math majors still frequently implemented the CNG principles, which proved facilitatory tools in the evolution of students' thinking about the eigenvalue–eigenvector relationships.

1. Background

1.1. Teaching and learning linear algebra

Since the mid-eighties, various researchers investigated the learning and teaching of linear algebra from an educational perspective (Dorier, 1991, 1995; Dreyfus & Hillel, 1998; Harel, 1985, 1987; Pavlopoulou, 1993; Robert & Robinet, 1989; Rogalski, 1994; Sierpinska, Dreyfus, & Hillel, 1999; Sierpinska, 1995; Uhlig, 2002). Hillel (2000) argued that students' difficulties in linear algebra are primarily proof related: "not understanding the need for proofs nor the various proof

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techniques; not being able to deal with the often implicit quantifiers; confusing necessary and sufficient conditions; making hasty generalizations based on very shaky and sparse evidence" (p. 191). With reference to Piaget and Garcia (1989) triad (intraoperational, interoperational, transoperational developmental stages), Hillel (2000) characterized linear algebra students' understanding of matrices and linear operators to be at the *interoperational* level of thinking – although they were expected to communicate at the *transoperational* level of thinking.

Harel (1989a, 1989b, 1990) found that geometric thinking is an undeniable prerequisite for students' understanding of linear algebra concepts. In 1990, educators across mathematics departments in the United States formed the linear algebra curriculum study group (LACSG) whose goal was to "initiate substantial and sustained national interest in improving the undergraduate linear algebra curriculum" (Carlson, Johnson, Lay, & Porter, 1993, p. 41). The LACSG recommended that linear algebra curricula place "strong emphasis on geometric interpretation;" consider "the needs and interests of students as *learners*;" and "utilize technology in the first linear algebra course" (Harel, 2000, pp. 177–179). LACSG further recommended that the core syllabus of a first course in linear algebra encompass matrix addition and multiplication, systems of linear equations, determinants, properties of R^n, eigenvalues and eigenvectors, orthogonality, and supplementary topics (Carlson et al., 1993, pp. 43–44). In particular, eigenvalue–eigenvector topics would include characteristic polynomial, algebraic multiplicity, eigenspaces, geometric multiplicity, similarity and diagonalization, symmetric matrices, orthogonal diagonalization, and quadratic forms (p. 44).

Harel (2000) developed a theoretical framework on which the teaching and learning of linear algebra in a technological environment with strong emphasis to geometry connections is based: "The core of this framework is three learning/teaching principles: the Concreteness Principle, the Necessity Principle, and the Generalizability Principle" (p. 180).[1] Harel and Kaput (1991), and Harel (1985, 1990) formulated the *concreteness principle*, as a fundamental approach for the teaching and learning of linear algebra, founded in Piaget's (1977) idea of conceptual entities. According to this principle, "for students to abstract a mathematical structure from a given model of that structure the elements of that model must be conceptual entities in the student's eyes; that is to say the student has mental procedures that can take these objects as inputs" (Harel, 2000, p. 180). Concreteness principle requires that "students build their understanding of a concept in a context that is concrete to them" (p. 182). He recommends MATLAB as a tool that would help students visualize vectors and matrices as concrete mathematical objects, in accordance with the concreteness principle.

Aligned with the *theory of problématique* (Balacheff, 1990; Brousseau, 1997), the *necessity principle* states that the learners "must see a need for what they are intended to be taught" (p. 185). Harel (2000) further emphasizes that the "need" should occur within the intellectual context, rather than social or economical contexts. He goes on to state that "the idea behind this [necessity] principle is that instructional environments must include appropriate constraints by which students can reflectively abstract mathematical conceptions and, at the same time, keep the situation at hand realistic" (p. 185). In this vein, "the notion of intellectual need is inextricably linked to the notion of epistemological justification: the learners' discernment of how and why a particular piece of knowledge came to be" (Harel, 2013, p. 119). A learner's engagement in a problem without a known solution method in advance (p. 119) could be thought of as the implementation of the necessity principle. Finally, the *generalizability principle*, which complements the concreteness and the necessity principles, states that "when instruction is concerned with a 'concrete' model, that is, a model that satisfies the Concreteness Principle, the instructional activities within this model should allow and encourage the generalizability of concepts" (p. 187). Harel (2000) further remarks that this principle should comply with the necessity principle. As an example, he argues that there is no intellectual need from a student's perspective in the generalization of the cosine of the angle between two vectors in 2D (or 3D) to that between two vectors in R^n (p. 188).

1.2. Literature review on eigentheory

Research studies taking into account cognition on eigentheory are scarce (Gol Tabaghi & Sinclair, 2013; Sinclair & Gol Tabaghi, 2010; Thomas & Stewart, 2011). In a research study framed in Tall's (2004) three worlds of mathematics (embodied, symbolic, formal) theoretical framework of cognitive development, Thomas and Stewart (2011) investigated university students' thinking processes associated with the equations $Ax = \lambda x$ and $(A - \lambda I)x = 0$. They identified "fundamental problems with student understanding of the definition of eigenvectors that lead to problems using it, and some of the concepts underlying the difficulties" (p. 275). Of particular interest, as observed by these researchers, was the fact that "how the second equation [$(A - \lambda I)x = 0$] is obtained from the first [$Ax = \lambda x$] implies a need to make it explicit in teaching, course books, and textbooks, explaining that the identity being used in the process is an $n \times n$ matrix, and it is the x that is being multiplied by this identity" (p. 295).

In a research study on mathematicians' conceptualization of eigenvalue–eigenvector relationships, Sinclair and Gol Tabaghi (2010) found that "mathematical meaning of eigenvectors depends strongly on both time and motion – hence, on physical interpretations of mathematical abstractions" (p. 223). These researchers further observed that gestures and motion-based conceptions played an important role in mathematicians' interpretations of the eigenvalue–eigenvector formalism. In particular, one of the participating mathematicians, JJ, established the eigenvector as "the hidden structure of the matrix that tells you what the matrix does for its living" (p. 231). In another study conducted with undergraduate students who

[1] The present report focuses on the "learning" aspect of these three principles.