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Students' understandings of multivariate integrals and how they may be generalized from single integral conceptions

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ABSTRACT

Prior research has documented several conceptions students have regarding the definite integral, but these largely describe how students understand single-variable integral expressions. There is little research that has focused specifically on investigating students' personal understandings of advanced types of integrals, including multiple integrals, nor how these might be generalized from the students' understanding of single-variable definite integrals. Through interview data from 10 students and survey data from 42 students, we present a variety of student understandings of multiple integrals. We also investigate the relationship between these understandings and students' conceptions of single-variable integrals.

1. Introduction

The undergraduate calculus series is a rich place for investigating the long-term development of students' understanding of mathematical ideas in that it takes core concepts such as functions, limits, derivatives, and integrals and progressively extends them from the single-variable context to the multivariate context to the more abstract contexts of real and complex analysis. This creates many instances in which students construct knowledge in first-semester calculus that is then increasingly generalized in subsequent courses. Some past studies have examined students' generalized knowledge of function (Kabael, 2011; Martinez-Planell & Trigueros-Gaisman, 2013; Weber & Thompson, 2014), domain and range (Dorko & Weber, 2014; Martinez-Planell & Trigueros-Gaisman, 2012), and derivative (Martinez-Planell, Trigueros-Gaisman, & McGee, 2014; Yerushalmy, 1997). While some research has touched on student understanding of multivariate integration (Jones, 2013), or on pedagogy related to multivariate integration (McGee & Martinez-Planell, 2014), no research has focused specifically on students' understandings of multivariate definite integrals, nor on how that understanding may be generalized from the single-variable context.

While the core concepts of functions, limits, and derivatives are all important topics for research in student understanding, we focus on the definite integral since it is less explored in the literature and is a particularly useful construct that is used frequently in pure mathematics (e.g., Brown & Churchill, 2008), physics (e.g., Serway & Jewett, 2008), engineering (e.g., Hibbeler, 2012), and other sciences (e.g., Lovell, 2004). Furthermore, many applications make use of more than simple single-variable integrals, but rather make use of multiple integrals, line integrals, Lebesgue integrals, or complex-valued

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integrals. In this study we focus on students' understanding of real-valued multiple integrals and how these understandings might be generalized from understandings of single-variable integrals.

In order to have a consistent terminology for the types of integrals we discuss in this paper, we will use the conventions of "single integral" to mean a definite integral whose integrand and differential are with respect to a single variable, and "multiple integral" generically to mean a definite integral whose integrand and differential(s) are in terms of two or more variables. A "double integral" refers to an integral whose integrand and differential(s) are in terms of exactly two variables, and a "triple integral" refers to an integral in terms of exactly three. Note that we define these in terms of the number of variables, not by the number of integral signs, since "$\int_R f(x, y)\,dA$" has only a single integral symbol, but is in terms of (presumably) two variables. While it is true that there are other versions of integration that could be considered, such as line integrals, Lebesgue integrals, or complex-valued integrals, we chose real-valued multiple integrals because they are the first encountered by students and many students finish their mathematics study prior to learning some of the more advanced types of integration. Furthermore, in this paper we focus on *definite* integrals as opposed to *indefinite* integrals since definite integrals tend to be the basis of most applications of integration. In summary, this paper is intended to shed light on the following two questions: (1) What understandings do students construct (whether stable or in-the-moment) for multiple integrals? (2) How can these understandings be seen as generalizations (or not) from previously documented student conceptions of single integrals?

2. Perspectives and definitions

2.1. In-the-moment versus stable understanding

In the previous section we have been careful to mostly use the word *understanding* as opposed to *conception*. Our reason for doing so is influenced by Thompson, Harel, and colleagues' recent work on distinguishing between *understanding/meaning in the moment* and *stable understanding/meaning* (Thompson & Harel, in preparation; Thompson, Carlson, Byerley, & Hatfield, 2014). In some of the past research documenting student conceptions (our own work included), a "conception" is often given the status of being a stable, permanent understanding a student might have for a particular mathematical object. However, an important implication of Thompson's and Harel's distinction for our present study is that a given student explanation provided during an interview might not fit this notion of "conception." Rather, it is possible, and probably quite common, for interviewed students to perform mental actions *in the moment* to create some kind of cognitive product, because the interviewer has asked them about something for which they have not already constructed a stable understanding. In this case, the student is creating an *in-the-moment* understanding of that mathematical object.

A *stable understanding* requires a cognitive state in which a scheme exists (Thompson et al., 2014), since the scheme provides the stability to make that understanding persist across time or location. In this sense, we use the word "scheme" to denote a mental organization that provides structure to a consistent way of interpreting a given mathematical object, or that provides a plan of mental action when encountering that mathematical object. One relationship between in-the-moment and stable understandings can be captured in the idea that "we construct stable understandings by repeatedly constructing them anew" (Thompson, 2013, p. 61). That is, a student's construction of the same *in-the-moment* understanding again and again may eventually lead to a *stable* understanding.

In this paper the constructs of *in-the-moment understandings* and *stable understandings* provide a language for describing whether a given student explanation shows evidence of the scheme-based stability of a stable understanding or whether the explanation suggests the student was constructing an in-the-moment understanding as a result of the interview setting. We consequently use the word *understanding* in this paper instead of "*conception*." Yet, because of established prior work, we continue the convention of calling student meanings for single integrals "*conceptions*." We do so since we do not attempt to perform this type of analysis on previous research on student understandings of integrals.

2.2. Students' conceptions of single-variable definite integrals

As mentioned, we make use of the word *conception* when discussing students' understandings for the single integral, due to the fact that past research has not distinguished between *in-the-moment* and *stable* understandings. Therefore, we use *conception*, generically, to mean how students might understand single integrals.

Since this study is designed to examine how understandings of multiple integrals may be seen as generalizations of single integral conceptions, we recount here some of the literature regarding how students may understand single integrals. While several studies have discussed difficulties students may have in making sense of or working with integrals (e.g., Bezuidenhout & Olivier, 2000; Orton, 1983; Wemyss, Bajracharya, Thompson, & Wagner, 2011), we are more interested in students' personal understandings of definite integrals. Along this vein, several studies have shown that there is considerable diversity in how students may conceptualize the definite integral (Czarnocha, Dubinsky, Loch, Prabhu, & Vidakovic, 2001; Hall, 2010; Jones, 2013, 2015b; Rasslan & Tall, 2002). Furthermore, students may have several distinct conceptions in their cognitive repertoire and may draw on them independently or in tandem when thinking about integrals (Jones, 2013). The conceptions discussed in this literature base are used to analyze how students' understanding of multiple integrals may be connected to and generalized from their understanding of single integrals. As such, brief descriptions of some of the conceptions described in previous studies are provided here.