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# Students' reasoning about relationships between variables in a real-world problem

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ABSTRACT

This paper reports on part of an investigation of fifteen second-semester calculus students' understanding of the concept of parametric function. Employing APOS theory as our guiding theoretical perspective, we offer a genetic decomposition for the concept of parametric function, and we explore students' reasoning about an invariant relationship between two quantities varying simultaneously with respect to a third quantity when described in a real-world problem, as such reasoning is important for the study of parametric functions. In particular, we investigate students' reasoning about an adaptation of the popular bottle problem in which they were asked to graph relationships between (a) time and volume of the water, (b) time and height of the water, and (c) volume and height of the water. Our results illustrate that several issues make reasoning about relationships between variables a complex task. Furthermore, our findings indicate that conceiving an invariant relationship, as it relates to the concept of parametric function, is nontrivial, and various complimentary ways of reasoning are favorable for developing such a conception. We conclude by making connections between our results and our genetic decomposition.



## 1. Introduction

This report is part of a larger study on second-semester calculus students' understanding of parametric functions (Stalvey, 2014). The concept of parametric function, much like a vector-valued function, can be thought of abstractly as a special relation from a subset of $\mathbb{R}$ to a subset of $\mathbb{R}^n$ for which each input, a real number, there is unique output, an ordered $n$-tuple. However, as with the general function concept (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994b), a more dynamic, intuitive approach is favorable, particularly when students are first introduced to the concept. Researchers have suggested ways of reasoning dynamically about curves defined parametrically (Keene, 2007; Oehrtman et al., 2008), but only a limited number of studies have investigated students' early experiences with the concept (Bishop & John, 2008). As parametric functions are often first introduced in the calculus sequence when students are studying different coordinate systems, we chose to focus our investigation on calculus students' reasoning.

An important component of understanding the concept of parametric function is the notion of reparametrization. A reparametrization of a curve changes the rate with respect to a parameter at which the curve is generated. In particular, an individual who understands the concept of parametric function is aware that, for a curve defined by $x = f(t)$ and $y = g(t)$, the relationship between $x$ and $y$ is invariant. Here, we borrow Thompson's (2011) use of the word *invariant* when he says, "the importance of covariational reasoning for modeling is that the operations that compose covariational reasoning are the very

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operations that enable one to see invariant relationships among quantities in dynamical situations" (p. 46). Mathematically, *invariant* means that the rate $dy/dx$ does not depend on the rates $dx/dt$ or $dy/dt$. In other words, we mean that the relationship between $x$ and $y$ remains the same when the rate, at which $x$ and $y$ vary, changes with respect to time.

One goal of our study was to capture students' ways of thinking about the reparametrization of a relationship between tangible quantities varying simultaneously in a real-world problem, in hopes to design curriculum that promotes the transfer of such thinking to purely mathematical contexts. In this paper, we address the following research questions:

1. How do students reason about relationships between variables in a real-world problem?
2. How do students conceive an invariant relationship between two quantities varying simultaneously with respect to a third quantity described in a real-world problem? In particular, how do students' ways of reasoning relate to the concept of parametric function?

## 2. APOS theory

Action–Process–Object–Schema (APOS) theory (Arnon et al., 2014; Asiala et al., 1996; Cottrill et al., 1996) is a constructivist framework for describing the cognitive development of mathematical concepts. In APOS theory, an *action* is a transformation of objects by reacting to external cues that give precise details on what steps to take. When an action is repeated, and the individual reflects on it, the action can be interiorized into a *process*. An individual who has a process conception can reflect on or describe the steps of the transformation without actually performing those steps. Additionally, new processes can be constructed by the means of reversal of a process or the coordination of two or more processes. When an individual becomes aware of the process as a totality and can perform additional actions or processes on it, then the process has been encapsulated into an *object*. Objects can be de-encapsulated to obtain the processes from which they came, which is often important in mathematics. The individual's collection of actions, processes, and objects organized in a coherent manner is his or her *schema* for a particular mathematical concept.

### *2.1. Genetic decomposition*

In APOS theory, a *genetic decomposition* is a hypothetical model for how an individual might construct his or her understanding of a particular mathematical concept. That is, it is a description of how an individual might construct the actions, processes, and objects that make up his or her schema for a concept. Below we offer a genetic decomposition for the concept of parametric function, elements of which will be illustrated when we discuss students' reasoning in Section 5 to demonstrate its viability.

Prior to the study of parametric functions, a student should have constructed schemas for $\mathbb{R}^2$ and function. The schema for $\mathbb{R}^2$ should include points as objects and curves as made up of points. The schema for function should include a process conception of single-variable, real-valued function that encompasses the notions of inverse function and function composition. The aforementioned constructions are comparable to those in the genetic decomposition for two-variable function offered by Trigueros and Martínez-Planell (2010) (see also Martínez-Planell & Trigueros Gaisman, 2012). The following are the constructions that we consider an individual should make in order to construct an understanding of the concept of parametric function:

1. Coordinate the processes of evaluating two functions $f$ and $g$ at values of $t$ to imagine $f(t)$ and $g(t)$ changing simultaneously as $t$ increases through values shared by Dom $f$ and Dom $g$. At this step, the individual can compare changes in $f(t)$ with changes in $g(t)$ over small intervals of $t$.
2. Coordinate the processes in Step 1 with the schema for $\mathbb{R}^2$ to construct elements of $\mathbb{R}^2$ defined by $(x, y) = (f(t), g(t))$. At this step, an individual with an object conception of point in $\mathbb{R}^2$ can describe the process of $(f(t), g(t))$ tracing out a curve in $\mathbb{R}^2$.
3. Suppose that one of the functions, say $f$, in Step 1 is invertible. Reverse the process of $f$ and encapsulate it into an object, namely $f^{-1}$. Then compose it with the other function $g$ in Step 1 to describe $y$ as a function $g \circ f^{-1}$ of $x$.
4. Encapsulate the process in Step 2 to compare and contrast congruent curves in $\mathbb{R}^2$ in terms of their points and the rate at which they are generated. At this step, the individual can conceive a relationship between $x$ and $y$ as invariant, namely the relation in Step 3.

We do not mean to suggest that the above steps make up the concept of parametric function in its entirety, but we assert that the presence of the above processes, which are the most relevant to the present paper, are essential to a process conception of parametric function. For an extended version our genetic decomposition, refer to Stalvey (2014).

## 3. Literature review

### *3.1. Function*

The concept of function has received a substantial amount of attention in existing literature (e.g., Breidenbach et al., 1992; Dubinsky & Harel, 1992; Dubinsky & Wilson, 2013; Oehrtman et al., 2008; Thompson, 1994b). Breidenbach et al.