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Provoking the construction of a structure for coordinating $n+1$ levels of units

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ABSTRACT

In considering mathematical development across multiple domains, researchers have implicated the critical role of an individual's ability to produce and coordinate units. Here, we describe a theoretically grounded instructional approach for promoting growth in units coordination. Our approach is informed by neuroscience, as well as existing research on units coordination. We present promising results from implementing our approach in a 10-week teaching experiment with a sixth-grade student named Cody. We demonstrate how, over the course of the 14 teaching sessions, Cody progressed from activity involving the coordination of two levels of units, to activity involving the coordination of three levels of units.

Integers, fractions, and algebra present notorious challenges for our students. Recent research on students' mathematical ways of operating indicates a common root cause. Namely, many students lack the ways of operating necessary to produce and coordinate the various levels of units involved in reasoning flexibly within each domain. With fractions, for example, these units include a unit fraction, the whole, and the composite unit formed by taking multiple unit fractions as a single quantity: For example, understanding 7/5 as a quantity (unit) composed of 7 units of 1/5, five of which constitute the whole (Steffe & Olive, 2010). A growing body of research indicates that the number of levels of units students can coordinate mediates opportunities for learning across several domains of mathematics (Ellis, 2007; Hackenberg, 2013; Hackenberg & Tillema, 2009; Izsák, Jacobsen, de Araujo, & Orrill, 2012; Olive & Çağlayan, 2008; Ulrich, 2012). This finding introduces a serious problem for mathematics educators because, even by middle school, few students can readily coordinate three levels of units (Boyce & Norton, 2015). Our study addresses this problem.

We conducted a teaching experiment to test a theoretically grounded approach to promoting growth in students' abilities to coordinate multiple levels of units. We used existing research on units coordination along with general findings from cognitive neuroscience to design sequences of instructional tasks. What kinds of growth might we promote over the course of the teaching experiment? What constraints might we experience in students' ways of operating during the tasks? This paper reports on results with a sixth-grade student during the 10-week teaching experiment. We begin with an overview of units coordination: what it is; why it is important; and how we might promote it.

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Fig. 1. Bars task.

Table 1
Stages of units coordination.

	Students' unit structures	Students' reasoning on the bars task
Stage 1	Students can assimilate with (take as given) one level of units, and may coordinate two levels of units in activity.	Students use the short bar to mentally segment the long bar, imagining how many times it would fit into the long bar. This activity might be indicated by head nods or sub-vocal counting.
Stage 2	Students can assimilate with two levels of units (a composite unit), and may coordinate three levels of units in activity.	Students mentally iterate the medium bar four times, with each iteration representing a 3. This activity might be indicated by the student uttering "3, 3, 3, and 3; 12."
Stage 3	Students can assimilate with three levels of units (a composite unit of composite units), and can thus flexibly switch between three-level structures.	Students immediately understand that there are four threes in the long bar. This assimilation of the task might be indicated an immediate response of "12", buttressed by an argument that 12 is four 3s.

1. What is units coordination?

From a neo-Piagetian perspective, building models of students' mathematics involves positing psychological structures that explain and predict students' *sensorimotor* (overtly physical, including verbal) activity when students are engaged in problematic situations (von Glasersfeld & Steffe, 1991). This observable activity becomes the basis for researchers to make inferences about students' *internalized* actions and the mental structures that might organize them. Once these internalized actions are organized within a structure for operating, we refer to them as *interiorized* actions, or *operations*. In addition to operational schemes (von Glasersfeld, 1995), structures include groupings of operations that organize the ways in which operations can be composed and reversed (Piaget, 1968, 1970). For example, children first produce the number 12 as the result of coordinated counting activity. Once their mental actions are organized as operations, 12 can become a composite unit (Steffe, 1992)—part of a structure in which 12 is composed of 12 units of 1.

Units coordinating refers to students' activity in producing units, including composite units. When this mental activity is interiorized, the actions become organized as operations within units coordinating structures. In particular, Steffe (1992) has described a kind of units coordination wherein a student distributes one composite unit over the elements of a second composite unit. Interiorization of this activity results in a structure of embedded units, as defined by the *distributing operation*. We use the following task to illustrate the various kinds of units students might produce and the operations they might use to coordinate them within units coordinating structures.

The student is given a short bar, a medium bar, and a long bar and is told that the short bar fits into the medium bar three times and that the medium bar fits into the long bar four times; then the student is asked to determine how many times the small bar fits into the long bar (see Fig. 1).

In addition to distributing, relevant operations may include *segmenting*, by which students mark off equal lengths of a given unit, and *iterating*, by which students produce a new unit from a number of connected copies of another unit (Steffe, 1992). Researchers use such constructs to explain observed activity and infer how students operate with units. In this paper, we refer to students who have interiorized actions for assimilating with n levels of units as operating at "Stage n" (where $n = 1, 2$ or 3). Note that assimilating with n levels of units involves the student interpreting a situation as an instance in which she can distinguish n levels of units and organize them within a units coordinating structure; the units and levels do not exist in the situation itself but in how the situation is interpreted. Table 1 outlines ways of operating associated with each level, along with typical responses to the bars task.

The three stages in Table 1 represent broad categories for operating, defined by the number of levels of units with which students can assimilate situations and the number of levels they can produce through activity. There is plenty of room for content-related growth within each stage, and researchers have offered finer-grained progressions within specific content domains. For example, Steffe (1992) describes two counting sequences—the *initial number sequence* and the *tacitly-nested number sequence*—that fit within Stage 1. Discussion of these sub-stages is beyond the scope of this paper, but interested readers can find an exquisite elaboration by Ulrich (2015). The broad stages defined in Table 1 align with the multiplicative concepts identified by Hackenberg (2010), except we frame them more generally to include content domains, such as integer addition (Ulrich, 2012), that may not involve multiplication.

We expect students to produce and coordinate units through activity first. Once students internalize that activity, they can begin to organize it within an assimilatory structure. Fig. 2 illustrates how such operations might be organized. In