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Proving as problem solving: The role of cognitive decoupling

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ABSTRACT

This paper discusses the process of proving from a novel theoretical perspective, imported from cognitive psychology research. This perspective highlights the role of hypothetical thinking, mental representations and working memory capacity in proving, in particular the effortful mechanism of *cognitive decoupling*: problem solvers need to form in their working memory two closely related models of the problem situation – the so-called *primary* and *secondary* representations – and to keep the two models *decoupled*, that is, keep the first fixed while performing various transformations on the second, while constantly struggling to protect the primary representation from being "contaminated" by the secondary one. We first illustrate the framework by analyzing a common scenario of introducing complex numbers to college-level students. The main part of the paper consists of re-analyzing, from the perspective of cognitive decoupling, previously published data of students searching for a non-trivial proof of a theorem in geometry. We suggest alternative (or additional) explanations for some well-documented phenomena, such as the appearance of cycles in repeated proving attempts, and the use of multiple drawings.

1. Introduction

Proving is sometimes considered a special case of problem solving (e.g., Mamona-Downs & Downs, 2005), hence models of mathematical problem solving can be useful for analyzing students' constructions of proofs. In particular, the various attributes of mathematical problem solving, as well as the stages and cycles that an individual may go through while solving mathematical problems (e.g., Carlson & Bloom, 2005; Schoenfeld, 1985), may be applied to the analysis of students' attempts to prove a mathematical theorem.

In analyzing and interpreting students' data (such as the appearance of cycles and the use of drawings), we are interested in moving from descriptive to explanatory models, and we have found it useful in this regard to consider more general cognitive mechanisms, in addition to the ones specifically dealing with mathematical thinking. In particular, we are interested in the extensive research by cognitive psychologists dealing with the uneasy relationship between intuitive and analytical thinking, and with the limitations of working memory. Highlighting the role of such mechanisms in the construction of proofs is the main goal of this paper.

We begin the theoretical introduction (Section 2.1) by surveying previous research on proving as problem solving. Next (Section 2.2) we introduce a novel theoretical framework, highlighting the role of *working memory capacity*, in particular the mechanism of *cognitive decoupling*, as studied by cognitive psychologists working in the areas of problem solving, decision

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making and reasoning (Kahneman, 2011; Stanovich, 2009). This framework is then illustrated (Section 2.3) by analyzing a common college-level scenario of introducing complex numbers.

In the main part of the paper (Section 3), we re-analyze from a cognitive decoupling perspective thinking-aloud protocols of two high-ability students trying to prove a non-trivial theorem in geometry, protocols which had previously been analyzed in terms of problem-solving phases and attributes (Koichu, 2004; Koichu, Berman, & Moore, 2007a). We identify several "struggles" of the problem solvers during the proving process, which depend on their ability to follow complicated imaginary scenarios, and to decide where to invest further effort. In these struggles, problem solvers need to form in their working memory two closely related models of the problem situation – the so-called *primary and secondary* representations – and to keep the two models *decoupled*, that is, keep the first fixed while performing various transformations on the second, all the while struggling to defend the primary representation from becoming "contaminated" by the secondary one.

The purpose of the re-analysis is to suggest further explanations for some well-documented phenomena, such as the appearance of cycles in repeated proving attempts, and the use of multiple drawings. Along with the more standard analysis in terms of the mathematically-specific problem-solving phases and attributes, the re-analysis highlights the role of some general cognitive mechanisms which are involved in the proving process.

2. Theoretical background

2.1. Related research on proving as problem solving

There is a growing number of studies on students' difficulties in constructing proofs, which employ the terminology and theoretical tools from research on problem solving (Furinghetti & Morselli, 2009; Koichu, Berman, & Moore, 2006; Weber, 2001, 2005). The standard problem-solving terminology, which originated in the seminal work of Pólya (1945/1973) and developed further in the eighties (Kilpatrick, 1985; Schoenfeld, 1985, 1992), describes the human struggle with solving non-routine mathematical problems in terms of phases and attributes.

A recent model that consolidates many earlier frameworks of mathematical problem solving (e.g., by Mason, Burton, & Stacey, 1982; Pólya, 1945/1973; Schoenfeld, 1985; Verschaffel, 1999) is offered by Carlson and Bloom (2005). The model postulates four problem-solving phases: *orientation, planning, executing and checking*. Embedded in the framework are two cycles, each of which includes at least three of the four phases. The model also includes a sub-cycle "conjecture—test—evaluate" and operates with various problem-solving attributes, such as conceptual knowledge, strategic or heuristic knowledge, metacognition, control and affect. The model emerged from a study in which research mathematicians were engaged in solving non-routine problems.

Similar phases and attributes appear in several studies on students' difficulties in constructing proofs. For instance, Weber (2001) studied students who already possess advanced knowledge of mathematical proof, and attributed their proving difficulties to the lack of strategic or heuristic knowledge, including domain-specific proving techniques, which theorems are useful and when, and when to use their procedural knowledge. Furinghetti and Morselli (2009) presented a study in which students' failures with proving were attributed to the interplay of many cognitive and affective factors. Specifically, in describing the proving process they took into consideration Pólya's four problem-solving phases, but also the work related to the cyclic nature of problem solving (Carlson & Bloom, 2005), the crucial role of the choice of representation (Boero, 2001; Simon, 1996), the presence of automatic sequential procedures (Barnard & Tall, 1997; Weber, 2001), the affective pathways (DeBellis & Goldin, 2006) and students' beliefs about mathematics (Schoenfeld, 1992).

Koichu (2004) and Koichu et al. (2006) have further developed the problem-solving terminology by introducing holistic categorizations of problem-solving behaviors. They identified four modes of heuristic behaviors[1] by focusing on the ways of utilizing and combining heuristics at different problem-solving phases. Specifically, the modes differ with respect to the heuristics used at the beginning of the solution process, the number of attempts to solve the problem, the number of different mathematical approaches involved in these attempts, typical combinations of problem-solving phases and local heuristics, and how the solver feels about her progress. Two of the four modes are particularly relevant to the present paper: circular and spiral.

Briefly, *circular heuristic mode* is characterized by numerous problem-solving attempts, which exceed considerably the number of different mathematical approaches involved in solving the problem. The orientation, planning and checking phases often overlap; some attempts do not include the executing phase. Usually the solver comes back to the approaches he or she tried before. In the *spiral heuristic mode*, the student does not just come back to her earlier approaches, but would further try to modify or combine them. Specifically, spiral heuristic behavior is characterized by numerous problem-solving attempts and a matching number of different approaches involved. New attempts continue and advance the previous ones, though occasional repetitions occur too. Orientation, planning and checking phases often overlap.

The two protocols discussed in this paper are initially analyzed in terms of the modes of heuristic behaviors, problem-solving phases and attributes. That analysis will serve as a baseline for the subsequent re-analysis, with particular attention to cognitive decoupling.

[1] The next two paragraphs is an abridged and slightly modified version of the description that appears in Koichu (2010).