Journal of Mathematical Behavior 38 (2015) 1–8

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Making sense of fraction quotients, one cup at a time

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ARTICLE INFO

Article history:
Available online 31 January 2015

Keywords:
Fractions
Products
Models
Representations
Elementary teachers

ABSTRACT

On the surface, we discuss a concrete case, in which a group of learners made sense of fraction division, beginning with a specific, concrete problem that demanded fresh insight. To meet its challenge, several education undergraduates, joined later by the authors, built mutable, evolving models to support their thinking. More fundamentally, we discuss those models in some depth, as foundations for a theoretical analysis of emergent sense and meaning. The initial problem begins, in effect, as a test case to investigate, but soon, with further understanding, it emerges as a special case, to support or represent an insight that will hold in general. The *kind of* knowledge to be built, therefore must shift. Hence, for us, the mathematics we discuss is mathematics *in the making*, anchored to evolving models, arguments, and explanations.

1. Introduction

To begin we treat a concrete case, in which several education undergraduates, for the first time, made sense of fraction division. We gave them a specific, concrete problem that required fresh insight. To meet its challenge, these learners, joined later by the authors, built mutable, evolving models to support their thinking. We discuss these models in some depth, as foundations for analysis of emergent sense and meaning.

Our analysis will build directly on two recent investigations. The first (Speiser, Walter, & Sullivan, 2007), from a theoretical perspective, centers on how specific case investigations can underpin emerging generality. The second (Speiser & Walter, 2011) explores whole number products, in part to illustrate how learners' models, understood simply as things that people build, discuss, and modify, support emerging proofs and explanations.

> "For us a model is a thing – no more, no less – a tool, designed, built, or imagined to help make sense of something that we seek to understand. Because whatever sense we make is our construction, the result of actions taken and considered over time, it follows that the models we discuss should be viewed as objects of reflection and design, hence as contingent, temporary, mutable, available for reconsideration, reconstruction, or rejection... Our analysis will emphasize what models can help people do." (p. 271)

Often we imagine creativity as the work of a special, isolated individual. Sometimes, certainly, it is. But here, instead, we explore how several people, in collaboration, address a challenge that impels them to extend and reshape what they know, can do, and understand (Speiser, Walter, & Maher, 2003). For this purpose we selected a proverbial tough nut to crack: fraction division, where mute submission to official rules might seem especially entrenched.

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http://dx.doi.org/10.1016/j.jmathb.2014.12.001
0732-3123/

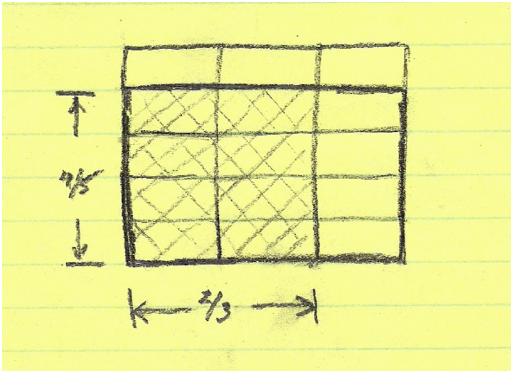

Fig. 1. A model for the operator product (2/3) × (4/5).

The shift from mimicking official practices to more thoughtful personal engagement (Speiser, Walter, & Lewis, 2004; Speiser et al., 2007) entails a change in how we understand ourselves as thinkers and as social actors. Building arguments from scratch can be a key experience, not just for oneself but perhaps especially in group collaboration. In prior work (Speiser, Walter, & Glaze, 2005) we documented how one learner, working with peers, first learned that mathematical ideas can offer helpful ways of thinking, rather than just templates to reproduce, fill in, and manipulate.

As insights emerge, the models we discuss will undergo important shifts. Such shifts might highlight insights, drawn from earlier experience, for further exploration and development. Hence, as we have emphasized (Speiser & Walter, 2011), a model in the kind of practice we discuss should be viewed, right from the outset, as a temporary sketch, available to reconsider, redesign, reject, or reinterpret.

Starting from a single, concrete problem, the models we discuss reflect more general concerns. The initial problem begins, in effect, simply as a test case (Speiser et al., 2007) to investigate. But soon, with further understanding, it emerges as a special case (loc. cit.), to support or represent an insight that will hold in general. The *kind of* knowledge to be built, therefore, has shifted. Hence, for us especially, the mathematics we discuss is mathematics *in the making* (Latour, 1987; Speiser & Walter, 2000), anchored to evolving models.

The work presented here took shape in an extended conversation, beginning in a college mathematics class for future elementary teachers. That conversation would continue, once the class had ended, through the writing that we now complete.

1.1. Setting, concepts, guiding questions

We begin with student thinking from a mathematics class[1] for future elementary teachers. The students, in groups of five or six, addressed key concepts through extended task investigations, where they built and tested explanations for solutions they had found. Further, they studied videos of young learners, from the Rutgers–Kenilworth longitudinal study (Davis, Maher, & Martino, 1992; Maher and Martino, 1992, 1996a, 1996b; Maher, Powell, & Uptegrove, 2010) who built, presented, and debated mathematically acceptable arguments. Our students, like the learners in these videos, sought explanations and supporting tools, in their own style and language, that would convince them (and us) on the spot, but also could potentially inform their future classroom work.

With our students, we agreed on a specific, formal view of multiplication (Speiser & Walter, 2011), that of operator products.[2] The first factor of an operator product, the *operator*, tells how many copies of the second factor, the *operand*, should be combined. For example, in the product 3 × 6, the operator, 3, counts groups of six. Students in a prior class (around 1999) extended this idea to fractions. The work that we consider here will build from that extension.

To be precise, consider 2/3 as an operator. To multiply by 2/3, our students would seek 2/3 of the given operand. They would find, for instance, (2/3) × (4/5), based on a model like that shown in Fig. 1.

Here one might, as shown, outline the area for 4/5 first, next draw further lines for 2/3, and finally shade 2/3 of the outlined area for 4/5. One explanation, for example, might proceed as follows. We see 3 × 5 = 15 small rectangles in the large rectangle, each worth 1/15.[3] Of these, 2 × 4 = 8 have been shaded, so the sought-for result must be 8/15. The reasoning presented here,

[1] Mathematics Education 306, Brigham Young University, January 2009, taught by the first author (Speiser & Walter, 2000, 2011; Speiser et al., 2004, 2007).

[2] This idea came from Robert B. Davis (1980), from his classic Postman Stories activity for signed integer arithmetic. We discussed this activity further, for the present classroom context, in Speiser and Walter (2011).

[3] Indeed, the array in Fig. 1 has three columns and five rows.