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Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts

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ABSTRACT

Research in mathematics and science education reveals a disconnect for students as they attempt to apply their mathematical knowledge to science and engineering. With this conclusion in mind, this paper investigates a particular calculus topic that is used frequently in science and engineering: the definite integral. The results of this study demonstrate that certain conceptualizations of the definite integral, including the area under a curve and the values of an anti-derivative, are limited in their ability to help students *make sense* of contextualized integrals. In contrast, the Riemann sum-based "adding up pieces" conception of the definite integral (renamed in this paper as the "multiplicatively-based summation" conception) is helpful and useful in making sense of a variety of applied integral expressions and equations. Implications for curriculum and instruction are discussed.

1. Introduction and rationale

In the last decade there has been increased attention given to researching students' understanding and use of the calculus topic of the definite integral (e.g., Bajracharya & Thompson, 2014; Jones, 2013; Kouropatov & Dreyfus, 2013; Rasslan & Tall, 2002; Sealey, 2014; Thompson & Silverman, 2008; Wemyss, Bajracharya, Thompson, & Wagner, 2011). Integration is a key topic that deserves our attention for several reasons. In a purely mathematical sense, it is a significant component in subsequent mathematics courses, including further coursework in the calculus series (Salas, Etgen, & Hille, 2006; Stewart, 2012; Thomas, Weir, & Hass, 2009) and in higher level mathematics, such as differential equations (Boyce & DiPrima, 2012) and complex analysis (Brown & Churchill, 2008). Thus, integration is an important foundational concept for a program of study in mathematics. However, the integral goes much further than this; it also serves as the basis for many real world applications in science and engineering. Physics and engineering textbooks regularly use integrals to define and compute natural phenomena like force, mass, center of mass, impulse, flux, circulation, energy, work, tension, and aspects of kinematics (Hibbeler, 2012; Pytel & Kiusalaas, 2010; Serway & Jewett, 2008; Wilson, Buffa, & Lou, 2010).

Yet this begs the question why we, as mathematics educators, should be concerned with how science disciplines, such as physics and engineering, use integration. The answer to this question is two-fold. First, as a "service course," first-year calculus at many universities is largely filled with students planning on majoring in science and engineering fields (Bressoud, Carlson, Mesa, & Rasmussen, 2013). Consequently, it may be that the students coming into our calculus classes are less motivated by pure mathematics than they are about being able to use the mathematics in their respective disciplines. Calculus instructors should be willing to address the needs of this large segment of their student population. Second, there is currently a push to

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improve science, technology, engineering, and mathematics (STEM) education as a connected whole (College Board, 2012; President's Council of Advisors on Science and Technology, 2012). This push implies that effort should be made by instructors of all STEM courses, including calculus, to assist in promoting success in these important fields of study.

Unfortunately, studies in both mathematics and science education give evidence of a serious disconnect between mathematics and the science disciplines it serves. For example, science contexts give additional layers of meaning to variables in mathematical expressions and equations, making the application of mathematics to science challenging (Dray & Manogue, 2005; Redish, 2005; Torigoe & Gladding, 2011). The way students learn mathematics in their mathematics courses does not always line up with how they need to draw on that knowledge in science and engineering (Gainsburg, 2006, 2007). Bridging that gap must involve insight from the mathematics education community; it is not a task for science education alone. We should develop mathematical understanding in our courses that supports application to other fields.

While we are beginning to learn how students create and hold knowledge about the definite integral (see Grundmeier, Hansen, & Sousa, 2006; Jones, 2013; Rasslan & Tall, 2002), there is scant evidence about how these cognitive constructs actually play out for students in making sense of the expressions and formulas in which integrals are present. Sealey and Engelke (2012) suggest that the area conception alone is not sufficient for robust understanding of integrals, and (Jones, 2013) supports this conclusion with an example of a student who struggled to interpret a physics integral through the area lens. Furthermore, Jones describes how a shift to an *adding up pieces* conception helped this student make sense of the integral he was struggling with. Neither of these studies, however, constitutes a deep analysis of how various conceptualizations of the definite integral provide students with the cognitive resources to understand integrals in mathematics and science contexts. Without such an analysis, it is difficult to determine how to provide students the opportunities to construct knowledge of the integral that will satisfactorily enable them to apply their knowledge to science and engineering.

This study attempts to provide some of this needed analysis by carefully examining how certain conceptualizations of the integral drive understanding in mathematics and science contexts. The study examines, in both of these contexts, student conceptualizations of the definite integral that are related to three common interpretations of the definite integral: (a) as the area under a curve, (b) as the values of an anti-derivative, and (c) as the limit of Riemann sums (see Salas et al., 2006; Stewart, 2012; Thomas et al., 2009). Note that even though the anti-derivative notion is often thought of as the province of *indefinite* integrals, students show a tendency to interpret *definite* integrals through the anti-derivative lens as well (Jones, 2013). Each of these three conceptualizations of the integral is evaluated for how helpful or useful it is for *making sense* of definite integral expressions and equations. Specifically, this paper attempts to shed light on the following three core questions: (1) Which of these three conceptualizations of the definite integral appear to be most useful for making sense of integrals in a pure mathematics context? (2) Which of these three conceptualizations of the definite integral appear to be most useful for making sense of integrals in an applied physics context? (3) What is the overlap or disjunction between these two contexts?

2. Background

2.1. Symbolic forms of the definite integral

This paper builds on a previous study (Jones, 2013) that details several conceptualizations of the definite integral held by calculus students. In this section, the reader is briefly acquainted with three of the conceptualizations described in that study, which deal with how students cognitively hold the familiar notions of area under a curve, anti-derivatives, and Riemann sums in connection with integrals. While there are certainly meanings beyond these three conceptions (e.g., Hall, 2010), these are the only ones under analysis in this study. The reason for choosing to concentrate only on the these three conceptualizations is that calculus textbooks often focus on area, anti-derivatives, and the Riemann integral during the exposition of and treatment of integrals (see, for example, Salas et al., 2006; Stewart, 2012; Thomas et al., 2009). Integrals in texts are often motivated by the study of irregular areas under the graphs of functions, approximated with finite Riemann sums, defined using the Riemann integral definition, and worked with using anti-derivatives. It is important to note that student conceptualizations are not necessarily *equivalent* to these ideas, but are rather *based* on them.

The way in which students cognitively possess these three conceptualizations is described through the lens of *symbolic forms* (Sherin, 2001). A symbolic form consists of a *symbol template* and a *conceptual schema*. The *symbol template* is the structure or arrangement of the symbols in the expression or equation, as in $\int_{[]}^{[]}[]d[]$ or $\int_{[]}[]d[]$, where each box "can be filled in with any expression" (Sherin, 2001, p. 490). The *conceptual schema*, on the other hand, "is the idea to be expressed" in those symbols (Sherin, 2001, p. 491). That is, it's the meaning that students see as being represented by the symbols in the definite integral structure. Three symbolic forms of the integral, called the *perimeter and area*, the *function matching*, and the *adding up pieces* forms, describe conceptualizations based on the area under a curve, anti-derivative, and Riemann sum conceptions, respectively. This paper does not analyze these symbolic forms in and of themselves, and the reader is referred to Jones (2013) for more detailed descriptions and an analysis of these cognitive structures. Rather, in this paper, these symbolic forms were used during the analysis for determining which conceptualization of the definite integral students were drawing on during a particular interview item. Therefore, it is necessary to briefly acquaint the reader with these three symbolic forms.