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On students' understanding of the differential calculus of functions of two variables

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ABSTRACT

APOS Theory is applied to study student understanding of the differential calculus of functions of two variables, meaning by that, the concepts of partial derivative, tangent plane, the differential, directional derivative, and their interrelationship. A genetic decomposition largely based on the idea of a directional slope in three dimensions is proposed and tested by conducting semi-structured interviews with 26 students who had just taken a course in multivariable calculus. The interviews explored the mental constructions of the genetic decomposition they can do or have difficulty doing. Results give evidence of those mental constructions that seem to play an important role in the understanding of these important concepts.

Introduction and purpose of the study

It is common that a quantity corresponding to an observable physical phenomenon will depend on several other quantities. Hence multivariable functions are a natural tool to use in modeling such phenomena. This in turn implies that the simplest case, the calculus of functions of two variables, is a tool of fundamental importance in the study of mathematics, engineering, and applied and social sciences. Students in these fields will undoubtedly need some mastery of this topic to describe observations and communicate within their own scientific discipline. These functions have been receiving increased attention in mathematics education research literature. In this article we present the results of a study of students' understanding of some of the main ideas of the differential calculus of functions of two variables. We start with a brief literature review.

The pioneering work of Yerushalmy (1997) considered the generalization involved in the transition from function of one variable to function of two variables, and stressed the importance of the interplay between different representations to generalize key aspects of these functions and to identify changes in what seemed to be fixed properties of each type of function or representation. Kabael (2009, 2011) argued that students' knowledge of the general function concept plays a fundamental role in their understanding of functions of two variables. She used the idea of the "function machine" as a cognitive root to generalize key notions of one-variable functions to two-variable functions and concluded it had a positive impact in student learning. She also reiterated the importance that conversions between different functional representations have on student understanding. In other related work, Montiel, Wilhelmi, Vidakovic, and Elstak (2009) considered student understanding of

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the relationship between rectangular, cylindrical, and spherical coordinates in a multivariable calculus course. They found that focusing on conversion among representation registers and on individual processes of objectification, conceptualization and meaning contributes to a coherent view of mathematical knowledge. Weber (2012) and Weber and Thompson (2014) considered student understanding of graphing functions of two variables from the point of view of quantitative and covariational reasoning. They propose a hypothetical learning trajectory that they argue will help students generalize their ideas of graphing from functions of one variable to functions of two variables. Dorko and Weber (2014) studied how students generalize the notions of domain and range from 2 to 3 dimensions mainly focusing on how the 2 dimensional constructs of the students influence their understanding of these ideas in 3 dimensions. Among other things they observe that some students lack flexibility in the use of variables, attaching the meaning of "domain" or "range" to the variable itself. They do not consider students' graphical representation of domain in three-dimensions, student understanding of restricted domains, or the formal notion of domain as a set of ordered pairs. Further along in this paper there will be a short discussion exploring some of the relationships between the previously mentioned article of Weber and Thompson (2014) with those of Trigueros and Martínez-Planell, which we now discuss.

The importance in converting between different representations stressed by most of the previously mentioned articles has also been recognized by our own previous and this present work on student understanding of functions of two variables. Trigueros and Martínez-Planell (2007, 2010) studied student geometrical understanding of functions of two variables and found, among other results, that students' mental construction of the schema for R^3, particularly their ability to predict and place properly in space the intersection of fundamental planes (planes of the form $x=c$, $y=c$, $z=c$, for a constant c) with surfaces plays a crucial role in their ability to graphically analyze these functions. It was found that many students show difficulty particularly when placing the intersection curve in its appropriate place in space and that this limits their ability to generalize and use properties of functions of one variable in the context of functions of two variables. Hence it is important to explicitly consider this issue during instruction since this is the geometric expression of the key idea allowing generalization from two to three dimensions, that is, to hold a variable fixed. Also pertinent to the present study is the observation that many students' have difficulty understanding formal aspects of the definition of function of two variables including identifying and geometrically representing the domain and the range, particularly when the domain is restricted (Martínez-Planell & Trigueros, 2009, 2012a). In this case the failure to generalize follows in part from students' lack of a complete understanding of domain and range of functions of one variable as well as from difficulties doing treatments and conversions within and between different representations. Hence instruction must promote actions that aid interiorization into corresponding processes for the domain and range of functions of two variables paying explicit attention to those that explore conversions between representations. This is relevant to the present study since as a consequence, these students may have further difficulty with the geometric representation of some aspects of the differential calculus such as the direction vector in a directional derivative, or the horizontal and vertical changes of a secant line in the definition of a partial derivative. Results from these studies led the same authors to design and analyze activities aimed at helping students make the mental constructions found to be difficult when learning functions of two variables, to class test them, and to conduct another round of student interviews to evaluate the results Martínez-Planell and Trigueros (2012b, 2013). These two articles Martínez-Planell and Trigueros (2012b, 2013) report on new observations that were made, some of which lead to further refinements of the preliminary genetic decomposition for functions of two variables and to improvements in the corresponding activity sets designed to help students make the conjectured mental constructions. It was observed that some students had not interiorized the action of point by point evaluation into a mental process, which ends up being particularly constraining in the case of functions of two variables which are described with only one variable (cylinders) and more generally in forming transversal sections and contours that result in a free variable. It was also observed that familiar algebraic expressions can act as cognitive obstacles restricting students' understanding. Overall, these articles have examined the graphical and formal understanding of functions of two variables. Most of these articles have applied APOS Theory together with semiotic representation theory. These articles provide a basis for our study of the differential calculus of functions of two variables.

The concept of variation in multivariable calculus has not received much attention in mathematics education research. Understanding the role of the differential calculus in the description of the behavior of functions of several variables is important both in advanced mathematics and in applications but little is known about how students' understand the basic concepts of partial derivatives, tangent planes, the differential, and directional derivatives. Some research on the differential calculus of functions of two variables has been conducted. Tall (1992, 2012) discussed the visualization of differentials in 3 dimensions providing a model for such visualization that is in accordance with the one we use for this as well as for planes, tangent planes, and the vertical change in a directional derivative. Thompson, Manogue, Roundy, and Mountcastle (2012) gave evidence of students' difficulties geometrically interpreting partial derivatives and mixed partials in the context of an application to another discipline. Weber (2012) included a discussion of the rate of change for functions of two variables focusing on the use of covariational thinking to help students build a notion of rate of change in space. This last paper's treatment of directional derivative centered on students' development of a symbolic representation and considered neither students' geometrical understanding of directional derivatives nor its relationship to other important ideas of the differential calculus of functions of two variables, as will be done in the present paper. Weber (2012) argued that a coherent understanding of rate of change of one quantity with respect to another relies on thinking about function as covariation of quantities and sustaining a process view of function. While we use another theoretical framework, our focus on the ideas of vertical change, horizontal change, their ratio and what it would mean to have a process conception of their ratio, makes our results consistent with the ideas presented by this researcher. Further observations comparing these two theories can