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Analyzing student understanding in linear algebra through mathematical activity

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ABSTRACT

In this paper we characterize students' conceptions of span and linear (in)dependence and their mathematical activity to provide insight into their understanding. The data under consideration are portions of individual interviews with linear algebra students. Grounded analysis revealed a wide range of student conceptions of span and linear (in)dependence. The authors organized these conceptions into four categories: travel, geometric, vector algebraic, and matrix algebraic. To further illuminate participants' conceptions of span and linear (in)dependence, the authors developed a categorization to classify the participants' engagement into five types of mathematical activity: defining, proving, relating, example generating, and problem solving. Coordination of these two categorizations provides a framework that proves useful in providing finer-grained analyses of students' conceptions and the potential value and/or limitations of such conceptions in certain contexts.

1. Introduction

The purpose of the study reported in this paper was to investigate student understanding of span and linear (in)dependence in linear algebra and to contribute to the body of knowledge regarding how individuals understand undergraduate mathematics. This study fits within a larger research program in which we explore students' transitions from informal to more formal ways of reasoning in linear algebra and leverage that research to produce curricular materials that promote a student-centered, inquiry-oriented approach to the teaching and learning of linear algebra. In particular, our research goals for the current study were (a) to classify students' conceptions of span and linear (in)dependence, and (b) to investigate how students use these conceptions to reason about relationships between span and linear (in)dependence. We first oriented our analysis of data from individual interviews through a grounded theory approach (Glaser & Strauss, 1967) in order to identify student conceptions of span and linear (in)dependence. We noticed that in coding students' conceptions, for which we made use of Tall and Vinner's (1981) construct of concept image, our analysis was facilitated by noting the type of mathematical activity in which the students were engaged as they shared their ways of reasoning. In other words, the interview question to which a student was responding had the potential of eliciting different aspects of the student's concept images. This is consistent with Vinner's (1991) notion of *evoked concept image*. For example, a student's reasons why a claim was true or false revealed ways of thinking about the associated concepts differently than did his or her response to "how do you personally think about this concept?" As such, we identified within the data set five mathematical activities in which students engaged during the interviews: defining, proving, relating, example generating, and problem solving. Within

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this paper we show how these mathematical activities can be used as a lens to further refine characterizations of students' concept images of span and linear (in)dependence.

Given these activity categories, our refined research objectives are (a) to investigate students' concept images of span, linear (in)dependence, and relationships between the two concepts and (b) to use the mathematical activities of defining, proving, relating, example generating, and problem solving in coordination with students' concept images in order to provide deeper insight into their understanding. Section 5 details the four concept image categories that grew out of our data: travel, geometric, vector algebraic, and matrix algebraic. We also define the five mathematical activities and provide examples of how coordination of the concept image categories with mathematical activity categories informed analysis of student thinking. Finally, we detail a framework of dual categorizations, provide an example using one student's response, and use this to provide richer descriptions of three students' understanding of linear independence, linear dependence, and span.

2. Theoretical perspective and literature review

The larger research program from which these data are drawn is framed by Cobb and Yackel's (1996) emergent perspective. From an assumption that that mathematical development is a process of active individual construction and a process of mathematical enculturation, this framework coordinates the individual cognitive perspective of constructivism (von Glasersfeld, 1995) and the sociocultural perspective based on symbolic interactionism (Blumer, 1969). Because the current research focuses on individual students' understanding within an interview setting, we restrict our analysis to the mathematical conceptions that individuals bring to bear in their mathematical work (Rasmussen, Wawro, & Zandieh, 2015). Within our analysis, we are guided by the assumptions that learners acquire knowledge from their daily experiences, that prior conceptions affect interaction with new ideas, and that knowledge structures are contextually dependent (diSessa, 1993). As such, we do not claim that our analyses of students' responses are the exact way that the participants thought about the concepts at the time of the interview; rather, we view their communication with the interviewer as data that acts as a proxy for how they think and reason about the mathematical content. This orientation to research aligns well with the use of Tall and Vinner's (1981) concept image framework, which facilitates our characterization of the nuanced ways in which individuals conceptualize mathematical ideas.

Given that the importance of linear algebra in the undergraduate mathematics curriculum because of both its wide applicability in the sciences and its pivotal role in the transition into more abstract and formal mathematics (Harel, 1989), the body of research regarding the teaching and learning of linear algebra has grown over the past few decades. The volume, *The Teaching and Learning of Linear Algebra*, edited by Dorier (2000), rises to the fore as a particularly influential collection of results in this research area. This volume includes empirical research regarding the "object of formalism" and teaching interventions that take this into account (Dorier, Robert, Robinet, & Rogalski, 2000; Rogalski, 2000), as well as various categorizations for modes of thinking and description in linear algebra (Hillel, 2000; Sierpinska, 2000). This latter work by Hillel (2000) and Sierpinska (2000) is of particular interest for this paper on analyzing student understanding of span and linear independence.

Hillel suggested three possible modes of description for vectors and vector operations, namely geometric, algebraic, and abstract. The *abstract mode* uses language of generalized theory, including terms such as dimension, span, linear combination, and subspace. The *algebraic mode* uses concepts more particular to the vector space $\mathbb{R}^n$, such as matrix, rank, and systems of linear equations. Finally, the *geometric mode* uses language that is familiar from our lived experiences, such as point, line, plane, and geometric transformation (p. 192). Hillel details difficulties students have within a given mode (such as confusion potentially caused by describing vectors as both arrows and points, both of which are a geometric description of vectors), as well as in moving between modes (such as how the difficulty in change of basis problems within $\mathbb{R}^n$ may relate to switching between algebraic and abstract modes). Connecting to this work, Larson (2010) noted that students often seem to blend these modes of representation. For instance, she gives examples such as "span of a matrix" or "linearly dependent matrix." These may arise from a blending of different modes (namely abstract and algebraic). This might also arise from students misattributing properties of a set of vectors, such as span and linear independence, to a matrix, although students may speak metonymically (Lakoff & Johnson, 1980) as if a single matrix is itself the set of vectors.

Attributing them to the historical development of linear algebra, Sierpinska (2000) suggests three modes of thinking and reasoning that coexist in linear algebra: synthetic-geometric, analytic-arithmetic, and analytic-structural. The first mode focuses on spatial reasoning, the second on algebraic manipulation and representation, and the third on formal, theorem-based and axiomatic thinking. The work of Dogan-Dunlap (2010) uses this framework to characterize students' descriptions of linear independence and dependence. By comparing student responses to a written assignment with one including a geometric component via an online dynamic graphing module, Dogan-Dunlap found 17 different categories of student thinking. The author further labeled the categories of student responses as either geometric or algebraic/arithmetic (the author uses "algebraic" interchangeably with "structural"), determining that 11 categories from across multiple question responses could be labeled as geometric. Dogan-Dunlap concluded that the online dynamic module facilitated students' development of geometric thinking and integration of multiple modes of thinking, noting, "the geometric representations in the presence of algebraic and arithmetic modes appear to help learners begin to consider the different representational aspects of a concept" (p. 2158).

A large portion of the research on student learning of span and linear independence in linear algebra (Aydin, 2014; Bogomolny, 2007; Ertekin, Solak, & Yazici, 2010; Kú, Oktaç, & Trigueros, 2011; Stewart & Thomas, 2010; Trigueros & Possani,