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Validating proofs and counterexamples across content domains: Practices of importance for mathematics majors

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ABSTRACT

Validating proofs and counterexamples across content domains is considered vital practices for undergraduate students to advance their mathematical reasoning and knowledge. To date, not enough is known about the ways mathematics majors determine the validity of arguments in the domains of algebra, analysis, geometry, and number theory—the domains that are central to many mathematics courses. This study reported how 16 mathematics majors, including eight specializing in secondary mathematics education, who had completed more proof-based courses than transition-to-proof classes evaluated various arguments. The results suggest that the students use one of the following strategies in proof and counterexample validation: (1) examination of the argument's structure and (2) line-by-line checking with informal deductive reasoning, example-based reasoning, experience-based reasoning, and informal deductive and example-based reasoning. Most students tended to examine all steps of the argument with informal deductive reasoning across various tasks, suggesting that this approach might be problem dependent. Even though all participating students had taken more proof-related mathematics courses, it is surprising that many of them did not recognize global-structure or line-by-line content-based flaws presented in the argument.

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1. Introduction

Proof and counterexample have been receiving an increasing level of attention in the mathematics education community because they play a critical role in the teaching and learning of mathematics. Given that proof and counterexample are fundamental to deepening individuals' learning and understanding in mathematics (Knuth, 2002; Peled & Zaslavsky, 1997; Schoenfeld, 2009; Yackel & Hanna, 2003), a wealth of research has investigated pre-college and college students' as well as secondary school mathematics teachers' abilities to understand and to generate proofs and counterexamples (e.g., Alcock & Weber, 2005; Bieda, Holden, & Knuth, 2006; Healy & Hoyles, 2000; Knuth, 2002; Peled & Zaslavsky, 1997; Selden & Selden, 2003; Weber, 2010). This body of research has also documented that secondary school students, undergraduate students, and secondary school mathematics teachers have considerable difficulty with proof and counterexample (e.g., Alcock & Weber, 2005; Bieda et al., 2006; Knuth, 2002; Peled & Zaslavsky, 1997; Selden & Selden, 2003; Weber, 2010). If secondary school mathematics teachers experience difficulty both in identifying and producing valid proofs and counterexamples, should it come as a surprise that students also have similar problems?

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In fact, mathematics teachers' content knowledge and beliefs about proof and counterexample affect the ways they implement proof-related tasks into classrooms, opportunities they provide for students to engage in proving and refuting, expectations they hold for students' learning, and judgments they make in students' arguments (e.g., Bieda, 2010; Stylianides, 2007; Stylianou, Blanton, & Knuth, 2009). Teachers' views on the role that proof can play in school mathematics influences proof-related tasks they identify from reform textbooks (Bieda, 2010). Teachers' mathematical understandings of what consists of acceptable justifications have an impact on how they provide feedback on students' justifications in classrooms (Bieda, 2010). In order to develop students' understandings of acceptable justifications in mathematics classrooms, teachers should possess sufficient content knowledge of proof and counterexample themselves. A focus on proof and counterexample validation may help teachers develop a better understanding of valid structures (e.g., proof by induction, proof by contraposition) and fine-grained definitions, theorems, and properties presented in arguments. These practices are equally important for undergraduate mathematics major for two reasons: (1) mathematics majors may become secondary school mathematics teachers and/or have opportunities to educate school mathematics teachers to carry out recent reform recommendations regarding proof and counterexample, and (2) mathematics majors should recognize the validity of their own work and of arguments presented in textbooks and lectures (Selden & Selden, 1995).

Recently, some researchers have paid attention to reading mathematical arguments and have reported undergraduate mathematics students' and mathematics teachers' performance in validating proofs (Alcock & Weber, 2005; Knuth, 2002; Selden & Selden, 2003; Weber, 2010). This body of research, however, tends to be based on a small sample of participants (Selden & Selden, 2003), a small number of assessment items (Alcock & Weber, 2005), or tasks in a single or two mathematical domains (Alcock & Weber, 2005; Knuth, 2002; Weber, 2010). Moreover, the aforementioned studies have primarily focused on participants taken from an introductory proof course (Selden & Selden, 2003; Weber, 2010). To date, little systematic data has been collected on how mathematics majors who have taken various advanced mathematics courses involving proof validate a variety of arguments in different mathematical domains—important practices for mathematics majors because they will be called upon to teach proof and counterexample across contexts in secondary school and undergraduate mathematics. To address the research gap, this study examines the strategies that mathematics majors use for validating various arguments in algebra, analysis, geometry, and number theory—the domains that are central in both high school and undergraduate mathematics. The results of this study give insight into how proof understanding develops as students gain mathematical expertise in a variety of courses at the undergraduate level.

Current reforms in U.S. mathematics education suggest that incorporating proof and counterexample into all content areas of the mathematical curriculum, from pre-kindergarten to undergraduate mathematics, is essential to support the development of students' mathematical reasoning (American Mathematical Society [AMS], 2001; Common Core State Standards Initiative [CCSSI], 2010; Mathematical Association of America [MAA], 2004; National Council of Teachers of Mathematics [NCTM], 2000, 2009). The CCSSI (2010) and the NCTM Standards (2000, 2009) further state that high school students should be able to understand proofs and produce both proofs and counterexamples. The AMS Conference Board (2001) suggested that future high-school mathematics teachers should develop adequate understandings of how to write formal proofs. The MAA Curriculum Guide (2004) recommended that undergraduate students should be able to identify proofs and counterexamples and proofs. In order to accomplish the aforementioned goals, undergraduate mathematics instructors and secondary mathematics teachers "should discuss the logical structure of the arguments that students present and assist students in critiquing each other's arguments" (NCTM, 2000, p. 346).

The limited literature investigating individuals' performance in validating arguments, however, show that some undergraduate students[1] and mathematics teachers[2] have trouble recognizing given arguments as valid mathematical proofs (e.g., Alcock & Weber, 2005; Goetting, 1995; Knuth, 2002; Selden & Selden, 2003; Weber, 2010). Knuth (2002) found that each teacher he studied rated at least one of the eight non-proofs as a proof, although 93% of them successfully identified the valid arguments as proofs. Goetting's (1995) results show that 15% of the undergraduate students accepted both a correct counterexample and an incorrect proof for the same statement. Selden and Selden (2003) reported that some undergraduate students tended to focus on superficial errors, such as algebraic expressions and symbolic manipulations, rather than global errors, such as proving the converse of the statement and the fundamental gaps in the given argument. Alcock and Weber (2005) and Weber (2010) reported similar findings when most undergraduate students in their studies did not succeed in validating proofs because they failed to check warrants used in the proofs.

The above findings should not come as a surprise, given that most U.S. undergraduate students spend considerable amounts of time observing and passively taking notes as professors present proofs in lecture-based courses (Weber, 2004). Because they just copy the final productions, they are less likely to notice the importance of the conceptual understanding of mathematics needed to develop proofs and counterexamples and to see proof and counterexample "as a tool for thinking more deeply about mathematics" (Stylianou et al., 2009, p. 4). In order to accomplish prevailing reform recommendations regarding proof and counterexample, it is important to create a learning environment where undergraduate students are engaged in validating proofs and counterexamples and discuss their determinations with others (Alcock & Weber, 2005; MAA, 2004; NCTM, 2000; Selden & Selden, 1995, 2003; Weber, 2008). To understand what environment can encourage undergraduate students to learn skills for validating arguments in different mathematical domains, we need to understand

[1] Undergraduate students in this paper refers to mathematics majors and pre-service secondary mathematics teachers.
[2] Mathematics teachers in this paper refers to in-service secondary mathematics teachers.