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A framework for example usage in proof presentations

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ABSTRACT

The use of examples has been identified as a pedagogical tool for proof presentations, however, the types and uses of examples in the context of proof presentations in advanced mathematics courses has not been described in detail. This study will organize the types of examples that were used in the proof presentations of four instructors into a coherent descriptive framework that is grounded in observation and interview data. This framework will present an integrated approach for when and how different types of examples are used during proof presentations.

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1. Introduction

In their advanced mathematics courses, undergraduate mathematics majors are expected to be able to comprehend mathematical proofs and write their own original proofs. This is well known to be a problematic shift for many students. In a traditionally taught course, attending to proof presentations in class is one of the primary ways in which students construct their understanding of what constitutes a proof (Weber, 2004). Because proving can be considered a type of problem solving, a proof presentation can be viewed as a model of a solution to a problem. Polya's (1945) book *How to Solve It* describes using examples to help students understand the problem or to lead them to a plan for solving the problem.

Throughout the literature on examples and proof, there are several different purposes for examples. They can be used in problem solving in general (Polya, 1945), determining the truth or falsity of claims (Alcock & Inglis, 2008; Lockwood, Ellis, Dogan, Williams, & Knuth, 2012), in constructing an original proof (Harel, 2001; Lockwood et al., 2012), or to aid in proof comprehension (Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012; Weber & Mejia-Ramos, 2011).

Some studies have argued that examples can be an obstacle for students' proving because of their tendency to adopt an empirical proof scheme (Harel & Sowder, 1998). Others suggest that examples play an important role in experts' conjecturing and proving, and thus students may benefit from using a variety of examples when exploring conjectures and proving (Inglis, Mejia-Ramos, & Simpson, 2007; Lockwood et al., 2012). This study contributes to this literature by investigating the types of examples used by instructors when presenting proofs in advanced mathematics courses and examining the pedagogical uses for the examples used by the instructors.

The research questions and results are restricted to the instructors' specific actions and the perceptions they expressed in the interviews. This study focuses on examples that are linked chronologically to the presentations of theorems and proofs. Interview data and inferences from the observation data were used to construct pedagogical uses for the types of examples. The specific research questions addressed in this study are:

1. What types of examples do instructors use in presentations of theorems and proofs in an upper-division proof-based mathematics course?

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2. When do these examples occur chronologically in relation to the presentation of theorems or proofs?
3. What are the instructors' pedagogical uses for the different types of examples when presenting the statement of a theorem or a proof?

2. Definition of "Example"

The term "example" has had varied meanings in the literature. Watson and Mason (2005) say that an "example" is "anything from which the learner may generalize" (p. 3). Their use of example is learner-dependent, allowing the learner to construct examples that may not be mathematically accurate. Others have taken the learner out of the picture and refer only to a mathematical requirement in their definition of example. Zazkis and Leikin (2008) use example to mean an "instance, illustration, case, or element of a mathematical idea, object, process, or class." Alcock and Weber (2010) use "example" in a much more restricted way, to mean "a mathematical object satisfying the definition of some concept."

In this study, I will consider a mathematical object an example if it has two properties: it must be specific and concrete as opposed to general and abstract. Specificity is a mathematical requirement; the object must represent a particular element of a larger class. Concreteness implies that students at this level must be able to either compute with or investigate properties of the mathematical object. Thus, concreteness is concerned with the accessibility of the mathematical object to the learner. Therefore, I will use the following definition: *An example is a specific, concrete representative of a class of mathematical objects, where the class is defined by a set of criteria.*

It is possible that a mathematical object can be mathematically specific but not concrete for a particular group of students. In an introductory analysis class, a function from the power set of the natural numbers to the real numbers can be defined by $f(A) = 0.a_1a_2a_3 \ldots a_n \ldots$ where $a_n = 0$ if $n \notin A$ and $a_n = 1$ if $n \in A$. This is a specific function, but most students at that level cannot investigate its properties because it is still too abstract to be accessible to them. However, if the instructor were to choose a specific, subset A, such as $A = \{1, 2, 3\}$ and show how $f(A) = 0.11100000\ldots$, then the object is much more likely to be concrete to the students and would therefore be classified as an example.

It is also possible that a mathematical object could not be specific enough to be classified as an example. In a presentation of a proof about the surjectivity of a composition of functions, an instructor in this study drew and labeled three "blobs" on the board to represent the three sets, arrows to represent the maps, and dots to represent elements of the set. Throughout the proof, he referred to the diagram. The diagram served as an alternative representation of the general proof and could not be said to specify a member of a class of functions with a given property. Therefore, it did not fit my definition of example and was said to be a *generic diagram*. This paper will investigate instructors' uses of examples, and therefore the generic diagrams that appeared in the data will not be included in the results.

Mathematical objects have multiple representations such as numerical, algebraic, or pictorial. In the same manner, examples may be represented in multiple ways. In this study, examples will not be classified by their representation, but rather according to their type and pedagogical use.

3. Literature review

This study is concerned with instructors' use of examples in their proof presentations in advanced mathematics lectures. Thus, I will first review current literature on teaching practices in advanced mathematics lectures. Next, I will review the literature related to use of examples. Examples play a crucial role for mathematics learning and teaching at all levels. A recent special issue of *Educational Studies in Mathematics* (Bills & Watson, 2008), highlights the importance of studying the role of examples for students, experts, and teachers. My review of the literature on examples will explore each of these three areas.

3.1. Teaching practices in advanced mathematics courses

Teaching practice refers to "what teachers do in and out of the classroom on a daily basis" (Speer, Smith, & Horvath, 2010). Although there have been many calls for studies addressing the teaching practices of university teachers (Harel & Fuller, 2009; Harel & Sowder, 2007; Speer et al., 2010), the literature contains very few responses. In particular, there has been very little research addressing the teaching practices of faculty members in upper-division proof based courses (Weber, 2004). Mejia-Ramos and Inglis (2009) performed a literature search in the top seven journals that have a history of publishing research in undergraduate mathematics education. They found 102 research papers addressing writing, reading, and understanding of proof by undergraduates, but none of the tasks in these papers were focused on proof presentation, either by instructors or by students. Recent studies have tended to focus on interviews with mathematicians on their pedagogical views of proof presentations (Alcock, 2010; Harel & Sowder, 2009; Hemmi, 2010; Lai, Weber, & Mejia-Ramos, 2012; Weber, 2011; Yopp, 2011). Although there have been some single case studies of classroom-based observations of how mathematicians present proofs (Fukawa-Connelly, 2012; Weber, 2004), the research in this paper contributes by examining four interrelated case studies of instructors' presentations of proofs, with an emphasis on how they use examples in their in-class proof presentations.