Journal of Mathematical Behavior 33 (2014) 119–130

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Reversible reasoning in fractional situations: Theorems-in-action and constraints

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ARTICLE INFO

Article history:
Available online 3 December 2013

Keywords:
Division
Fraction
Multiplicative reasoning
Reversibility
Units
Vergnaud's theory

ABSTRACT

The aim of this study was to investigate, at a fine-grained level of detail, the theorems-in-action deployed and the constraints encountered by middle-school students in reasoning reversibly in the multiplicative domain of fraction. A theorem-in-action (Vergnaud, 1988) is a conceptual construct to trace students' reasoning in a problem solving situation. Two seventh grade students were interviewed in a rural middle-school in the southern part of the United States. The students' strategies were examined with respect to the numerical features of the problem situations and the ways they viewed and operated on fractional units. The results show that reversible reasoning is sensitive to the numeric feature of problem parameters. Relatively prime numbers and fractional quantities acted as inhibitors preventing the cueing of the multiplication–division invariant, thereby constraining students from reasoning reversibly. Among others, two key resources were identified as being essential for reasoning reversibly in fractional contexts: firstly, interpreting fractions in terms of units, which enabled the students to access their whole number knowledge and secondly, the unit-rate theorem-in-action. Failure to conceptualize multiplicative relations in reverse constrained the students to use more primitive strategies, leading them to solve problems non-deterministically and at higher computational costs.

1. Background

The inspiration of this investigation stemmed from Piaget's concept of reversibility of thought and has been motivated by the recent plea made by Lamon (2007) for analyzing such a process: "Researchers know very little about reversibility or about multiplicative operations and inverses, and these could be subjects for a valuable microanalysis research agenda" (p. 661). Another theoretical thrust to conduct this research is that, in contrast to additive situations (Carpenter & Moser, 1983; Fuson, 1992; Nesher, Greeno, & Riley, 1982), reversibility has not been given much attention in the multiplicative domain. Since Piaget introduced the concept of reversibility, researchers working in different areas (Adi, 1978; Behr & Post, 1992; Dreyfus & Eisenberg, 1996; Krutetskii, 1976; Lamon, 1994; Olive & Steffe, 2002; Tzur, 2004; Vergnaud, 1988) have been evoking this idea but it has not been a major focus. Only one pertinent study related to reversible reasoning in the domain of fractions (specifically focused at the middle-school level) could be identified from the literature, namely Hackenberg (2005). In a previous study (Ramful & Olive, 2008), we attempted to characterize the ways in which students reason reversibly in a proportional situation. The current study focuses on reversible reasoning in the domain of fractions.

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http://dx.doi.org/10.1016/j.jmathb.2013.11.002

Piaget (1970) conceptualized reversibility as one of the four characteristics of an operation, the other three being regarded as follows: (i) as an internalized action (i.e., an action that can be carried out in thought as well as executed materially), (ii) as involving conservation/invariance (e.g., in the case of addition, we can transform the way we group together 5 + 1, 4 + 2 or 3 + 3 but what is invariant is the sum) (iii) as related to a system of operations in a structure. He characterized reversibility in two different forms: negation and reciprocity. Negation expresses the idea that every direct operation has an inverse which cancels or negates it. For example, addition is canceled by subtraction and multiplication by division. Piaget (1970) argued that "subtraction is simply the reversal of addition – exactly the same operation carried out in the other direction" (p. 22). Further, in terms of changes of position, he stated that negation involves the understanding that a movement in one direction can be canceled by a movement in the opposite direction. On the other hand, reciprocity deals with relational structures. For example, understanding the relation embodied in the equation 2 + 2 = 4 as a whole collection of 4 objects as well as two partial collections of two objects each (Chapman, 1992). Reversible reasoning is required to conceptualize the assimilation of the parts to the whole and to view the whole as consisting of parts. Such an additive part-whole schema represents a major conceptual achievement in the early school years (Resnick, 1992). The part-whole schema can also be observed in the multiplicative domain as is the case in the present study. For example, in a situation such as "if 2/5 of a parking lot holds 30 cars, how many cars can the parking lot contain?", one is working with the part-to-whole relation between a fraction and its referent whole (i.e., given a part, make the whole) instead of the whole-to-part relation (i.e., given the whole, make a part). In essence, Piaget used the term reciprocity to refer to the coordination between the 'two-sidedness' of a relation. For example, if $a > b$, then $b < a$ or if $a + b = c$, then $a = c - b$ or if $a = b$, then $b = a$.

Adi (1978) used the concept of negation and compensation to study the relationship between college students' developmental level and their performance on equation solving. She provided the equation $14 - (15/(7 - x)) = 9$ to illustrate her interpretation of negation and compensation. In solving this algebraic equation, negation is involved when one is asked to make the following inferences: 'Fourteen minus what equals nine?', 'Fifteen divided by what equals five?', and 'Seven minus what equals three?'. On the other hand, compensation is involved when one multiply both sides of the equation by $(7 - x)$ to obtain $98 - 14x - 15 = 63 - 9x$. Further, compensation occurs when one adds $14x$ to both sides of the equation to obtain $83 = 63 + 5x$ after which 63 can be subtracted to yield $20 = 5x$. In the final step, compensation is involved when one divides both sides by 5 to get the answer as $x = 4$.

Steffe and colleagues (e.g., Olive & Steffe, 2002; Tzur, 2004) conceptualized reversibility on the basis of their notion of a scheme as a three-part structure (situation, activity, and result). They referred to reversible reasoning when a result of scheme was fed back into the situation that generated the result. A person having a *reversible fraction scheme* can construct a fraction from a given whole and a whole from a given part. Following their research orientation, Tzur (2004) defined a reversible fraction conception as "the learner's partitioning of a non-unit fraction (n/m) into n parts to produce the unit fraction ($1/m$) from which the non-unit fraction was composed in the first place" (p. 93). Steffe and colleagues described both reversibility of operations and reversible schemes. Steffe (1992) also asserted that not all schemes are reversible. Another form of reversible reasoning can be inferred from Thompson and Saldanha (2003) who considered a fraction as consisting of two quantities that are in *reciprocal relationship of relative size*. For instance, amount A is 1/5 the size of amount B means that amount B is 5 times the size of amount A. Hackenberg (2005) used the term *reciprocity* to refer to the conceptualization of a quantitative relation as bi-directional where one can appropriate any of the two quantities in the quantitative relationship as the basis by which another quantity is produced. For instance, it entails knowing that if A is two-thirds of B, then B must be three-halves of A. Such a simultaneous conceptualization requires the construction of a reciprocal multiplicative relationship. Grounding her study on schemes, Hackenberg (2010) gave evidence to show how one of her participants had a uni-directional scheme, capable of inversion only while the other participant could work with reciprocal relationships through compensation. While Steffe and colleagues have used scheme theory and unit coordination to explain reversibility, the current study uses Vergnaud's theory (1988, 1994, 1996, 1997, 1998) to give an account of reversible reasoning.

In the current study, I analyze the ways in which two students construct one whole from a given fractional amount and correspondingly determine its measure. Behr and Post (1992) use the term 'construct-the-unit' to refer to such fractional problems in which students are required to construct the unit whole from a given fractional part in either discrete or continuous contexts. For instance, a typical problem in my study reads as follows: "Candy bar A is 5 units long. Its length is 3/4 of candy bar B. What is the length of candy bar B?" In so doing, I analyze the ways in which the students articulate the multiplicative comparison relationship between two quantities to find the measure (i.e., the length of candy bar B).

Thinking about a known amount that is n times as large as an unknown amount is the precise point where one is required to reverse his/her thinking to deduce that the unknown amount is $1/n$ times as large as the known amount. To solve these problems one has to use the inverse relationship between multiplication and division and carry out an operation of thought (that is, apply this inverse transformation). In other words, before doing the arithmetic operation, the mental operation of inverting the transformation multiplication to division must be carried out in order to connect the multiplicative situation with a divisive situation. I have interpreted reversible reasoning in this study as the construction of one whole from part of a whole or a quantity larger than a whole. Operationally, reversible reasoning is involved in deducing that if quantity A is m times as large as quantity B, then quantity B is $1/m$ times as large as quantity A. Such form of reasoning may occur at different points in the construction of one whole.

As pointed out by Carpenter, Fennema, Franke, Levi, and Empson (1999), the multiplicative comparison of two quantities results in a non-identifiable quantity. For instance, if bar A is 5 units long and bar B is 3 units long, then bar A is 5/3 as long