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Problems and solutions in students' reinvention of a definition for sequence convergence

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ABSTRACT

Little research exists on the ways in which students may develop an understanding of formal limit definitions. We conducted a study to (i) generate insights into how students might leverage their intuitive understandings of sequence convergence to construct a formal definition and (ii) assess the extent to which a previously established approximation scheme may support students in constructing their definition. Our research is rooted in the theory of Realistic Mathematics Education and employed the methodology of guided reinvention in a teaching experiment. In three 90-min sessions, two students, neither of whom had previously seen a formal definition of sequence convergence, constructed a rigorous definition using formal mathematical notation and quantification equivalent to the conventional definition. The students' use of an approximation scheme and concrete examples were both central to their progress, and each portion of their definition emerged in response to overcoming specific cognitive challenges.

1. Introduction and research questions

Developing facility with formal limit definitions is challenging for many calculus and introductory analysis students (Cornu, 1991; Cottrill et al., 1996; Fernandez, 2004; Tall, 1992; Tall & Vinner, 1981; Vinner, 1991; Williams, 1991). Successful interventions reported in the research literature have only occurred after significant effort to establish a robust conceptual understanding of various types of limits from which students may subsequently construct more rigorous formulations (Cory & Garofalo, 2011; Roh, 2008; Swinyard, 2011; Swinyard & Larsen, 2012). Insights into the nature of understanding sufficient for students to successfully reason about formal limit definitions are clearly valuable to both establish and achieve instructional goals for preparing students for more advanced coursework. These insights may also provide guidance for improving instruction in introductory analysis, often an important transition course in undergraduate mathematics programs. Even in a calculus course that does not aim to provide a rigorous treatment of limit definitions and proofs, developing a conceptual understanding sufficiently robust to support such formalization can improve the foundations of productive reasoning throughout calculus itself.

Oehrtman (2008) proposed an approach to developing the central concepts in introductory calculus that leverages conceptually accessible ideas about approximations and error analyses but also engages students in a coherent and rigorous treatment of the mathematical structure of these central concepts. Although developing facility with formal definitions is not a primary goal in Oehrtman's instructional framework, we hypothesize that its fidelity to the mathematical structure of

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limits may lay a strong conceptual foundation for formal definitions and arguments. With the aim of better understanding how students develop productive reasoning about sequence convergence and in an effort to capitalize on the approach proposed by Oehrtman, we recruited two students for a six-session guided reinvention teaching experiment from an introductory calculus course that employed the instructional framework. Neither student had previously seen the conventional $\varepsilon - N$ definition for sequence convergence. We situated the study toward the end of a second-semester calculus course so that students may have developed a robust concept image of convergence from which to abstract a formal definition. Over the course of the first three 90-min sessions of the teaching experiment, the two students worked under the guidance of two facilitators to reinvent a formal definition. The following research questions guided our work:

1) What problems do students encounter while reinventing the formal definition of sequence convergence, and how do students resolve these problems?
2) How does students' engagement of problems during a guided reinvention impact their understanding of their evolving definition?
3) To what extent and in what ways are students able to leverage reasoning about approximation and error analyses in their reinvention of the formal definition?

To provide context for the reader, we begin by situating this study within the existing literature on students' understanding of convergence. We then outline Oehrtman's instructional framework for introductory calculus. Following this, we detail the theoretical perspectives which framed the study, as well as the methodological design and analytic approaches employed. We then trace the evolution of the two students' definition of sequence convergence, highlighting the challenges the students experienced, as well as how they resolved those challenges.

2. Students' understanding of formal limit definitions

Existing research on students' understanding of limits has focused on the difficulties first-year calculus students experience as they initially encounter the concepts (e.g., Bezuidenhout, 2001; Davis & Vinner, 1986; Ferrini-Mundy & Graham, 1994; Monaghan, 1991; Tall & Vinner, 1981; Williams, 1991). This literature has established a litany of misconceptions students possess, such as viewing a limit as a boundary that cannot be reached, a boundary that can be reached but not surpassed, and the last term in an infinite sequence. Our aim is not to avoid these misconceptions, but rather we suggest that students resolving challenges derived from their initial intuitive conceptions is a natural stepping stone toward more rigorous understanding of mathematical concepts. Until recently, little was known about how students might make such progress. Cottrill et al. (1996) developed a seven-step genetic decomposition for how a student might construct a formal understanding of the limit of a function. Unfortunately, due to the absence of subjects exhibiting a complete understanding of the formal definition, the final three steps of this model were limited to conjecture rather than empirical evidence. More recently, Swinyard (2011) conducted two teaching experiments with pairs of students who had not previously encountered the formal $\varepsilon - \delta$ definition of the limit of a function, working to "reinvent" the definition by testing iterated refinements against sets of examples and non-examples. Swinyard and Larsen (2012) identified two critical cognitive shifts for students to transition from an informal to a formal understanding of limit: (i) shifting from a domain-first perspective used to *find* limit candidates to a range-first perspective used to *validate* limit candidates, and (ii) condensing the dynamic limiting process into a self-contained entity, independent of its individual stages.

Lakoff and Núñez (2000) outlined a series of metaphorical maps characterizing the structure of experts' understanding of the formal definition of sequence convergence. As Oehrtman (2009) pointed out however,

> The formal structures that are targets of these mappings evolved as mathematicians resolved a series of specific technical problems, but typical calculus instruction does not expect students to develop similar solutions or even to be engaged in such inquiry. Instead, students' initial concepts about limits are structured by nontechnical experiences, variations of which lead to significant idiosyncrasies in the purpose and structure of their spontaneous reasoning (p. 399).

Our goal in a guided reinvention is to engage students in reflecting on their current conception of limits to recognize and resolve problems with that conception. By iterating this process, students are meant to establish a rigorous and personally meaningful definition. Oehrtman's (2008) instructional framework proposes one way to help introductory calculus students establish a coherent and mathematically sound foundation that may support such a reinvention.

3. An instructional framework based on approximations and error analyses

Oehrtman's (2008) instructional design emerged from prior research to provide detailed characterizations of the structure of students' spontaneous reasoning about limit concepts and implications for their reasoning about other concepts defined in terms of limits (Martin & Oehrtman, 2010a, 2010b; Oehrtman, 2003, 2009; Sealey & Oehrtman, 2005). Extending Williams' (1991, 2001) characterization of students' base models, or metaphors, for reasoning about limits, Oehrtman (2009) analyzed 120 students' verbal and written statements about limits while solving non-routine problems. He identified five "strong" metaphors for limits that were both widely employed (emphatic) and significantly influenced students' reasoning