Journal of Mathematical Behavior 33 (2014) 209–229

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Conceptual blending: Student reasoning when proving "conditional implies conditional" statements

Michelle Zandieh*, Kyeong Hah Roh, Jessica Knapp

Arizona State University, United States

ARTICLE INFO

Article history:
Available online 31 December 2013

Keywords:
Proving
Conceptual blending
Conditional statements
Geometry
Key idea
Undergraduate mathematics education

ABSTRACT

Conceptual blending describes how humans condense information, combining it in novel ways. The blending process may create global insight or new detailed connections, but it may also result in a loss of information, causing confusion. In this paper, we describe the proof writing process of a group of four students in a university geometry course proving a statement of the form conditional implies conditional, i.e., $(p \to q) \Rightarrow (r \to s)$. We use blending theory to provide insight into three diverse questions relevant for proof writing: (1) Where do key ideas for proofs come from?, (2) How do students structure their proofs and combine those structures with their more intuitive ideas?, and (3) How are students reasoning when they fail to keep track of the implication structure of the statements that they are using? We also use blending theory to describe the evolution of the students' proof writing process through four episodes each described by a primary blend.

1. Introduction

In this paper we illustrate the power of the theory of conceptual blending to clarify issues that students have in proving a statement having the overall structure of a conditional implies a conditional, i.e., $(p \to q) \Rightarrow (r \to s)$. This logical structure occurs often in statements to be proven at the university level. For example, since the definition of A is a subset of B $(A \subseteq B)$ is a conditional statement ($x \in A$ implies $x \in B$), then a simple set theory statement such as "If $A \subseteq B$, then $A \cup B \subseteq B$," has this logical form. Another instance occurs when proving the induction step in a proof by induction. Likewise, a calculus statement such as "if a function f is increasing, then f is one-to-one" also has this logical structure because the definition of increasing (if $x_1 < x_2$, then $f(x_1) < f(x_2)$) and the definition of one-to-one (if $f(x_1) = f(x_2)$, then $x_1 = x_2$) are conditional statements. In addition, this structure can be found in real analysis contexts such as proving "every convergent sequence is Cauchy."

Although statements of the form "a conditional implies a conditional" are frequently used in mathematics, students often have difficulty proving properties stated in this form. These difficulties appear in the literature when the tasks given to students involve statements with the overall structure of a conditional implies a conditional (e.g., Knapp & Roh, 2008; Selden & Selden, 1995; Weber, 2001). However, these studies neither address students' understanding of the structure nor account for student difficulties with the structure when constructing a proof, which is a focus in this study.

In this paper we examine a group of students while they try to prove one direction of the equivalence of two forms of the parallel postulate of Euclidean geometry, that is implicitly structured in the form of a conditional implies a conditional. While analyzing and describing the students' proving activity, we will employ the theory of conceptual blending. As described below in the results section, blending theory allows us to illustrate connections across a number of different issues that occur

* Corresponding author at: Science and Mathematics Faculty, School of Letter and Sciences, Arizona State University at Polytechnic Campus, 6073 S. Backus Mall, Wanner 101M, Mesa, AZ 85212, United States. Tel.: +1 480 727 5014.
E-mail address: zandieh@asu.edu (M. Zandieh).

0732-3123/$ – see front matter © 2013 Published by Elsevier Inc.
http://dx.doi.org/10.1016/j.jmathb.2013.11.007

in proving and to organize the story-line of the evolution of student thinking in ways that are unique in the mathematics education literature.

2. Literature on proving

We define proving as the activities involved in constructing a proof which include the reasoning involved in such activities. Several studies indicate that undergraduate students often struggle with proving (e.g., Moore, 1994; Selden & Selden, 2008). Students' misunderstanding of logical rules and misinterpretation of logical statements may result in their difficulty with structuring their proofs (Brown, 2003; Harel, 2001). In particular, undergraduate students often have difficulty properly using the contrapositive equivalence (Stylianides, Stylianides, & Philippou, 2004), Modus Tollens (Inglis & Simpson, 2008), and other conditional inferences (Durand-Guerrier, 2003; Rettke, 2004). Many students tend to structure their proofs in terms of chronological order of their thought process instead of rearranging it with careful consideration of proper implications (Dreyfus, 1999). Alcock and Weber (2005) found that many real analysis students evaluate an argument as a valid proof as long as each assertion in the argument is valid even though the statements listed later in the proof do not follow legitimately from earlier assertions in the proof. The studies suggest that students often have difficulty with many aspects of structuring a proof.

Epp (2003) suggests that without explicit guidance about the mathematical meaning of conditionals, biconditionals, and quantified statements, undergraduate students might interpret these statements colloquially. Other studies support this claim by reporting students' mis-interpretation of such logical statements, especially conditional statements (Durand-Guerrier, 2003) and quantified statements (Dubinsky & Yiparaki, 2000; Roh, 2010). Furthermore, Selden and Selden's (1995) study discusses how students' lack of ability to identify hidden quantifiers and implications in informal statements may hamper their ability to determine a proof framework for the statement.

The literature also shows that students are often unable to bring useful syntactic knowledge to mind. Such knowledge includes formal definitions (Knapp, 2006) as well as theorems and properties (Weber & Alcock, 2004) of the mathematical concepts. Weber (2001) found that undergraduate students are often unable to strategically choose theorems when constructing a proof, even though the students already had knowledge of the theorems and possessed an accurate conception of mathematical proof. Indeed, students tend to predominantly use only a single mode of knowledge when constructing a proof (Alcock & Simpson, 2004).

Likewise, research calls attention to various forms of personal knowledge of mathematical concepts. Such knowledge is internally meaningful to an individual student (Pinto & Tall, 2002; Vinner, 1991). It helps a student recall conceptual ideas to apply when attempting to construct a proof. For instance, Knapp and Roh (2008) illustrate how advanced calculus students' ideas of convergence play a role in their attempt to prove every convergent sequence is a Cauchy sequence. Because of its private and informal nature, students' personal knowledge is often insufficient for them to know how to get started on a proof (Moore, 1994). Raman (2003) suggested the *key idea* as a means of connecting personal intuitive ideas and procedural knowledge when constructing a proof. She explains when students possess a key idea for a proof it gives them conviction as well as the basis for the formal mathematical proof. The key idea thus provides students with an important insight about relationships between ideas in the conceptual setting from which they may begin to build a more structured proof. Raman and Weber (2006), Raman Sundström and Zandieh (2009), and Zandieh, Larsen, and Nunley (2008) have illustrated ways in which a key idea was successfully used to bridge the gap between students' personal ideas for a proof and a more formal mathematical form of that proof.

The conditional implies conditional structure is common in statements proved by students of this level. However, researchers have not highlighted the role of the conditional implies conditional structure in their analysis, even though their students were proving statements of this type (e.g., Brown, 2003, 2008; Knapp & Roh, 2008; Selden & Selden, 1995). Thus we add to the literature by describing students' proving using this logical structure that is common in mathematical problems, but which has not been directly addressed in previous work.

To describe student proving we focus both on semantic issues of creating a key idea for a proof and syntactic issues of structuring the proof and working with the logical structure of the statements involved in the proof. To do so we draw on the theory of conceptual blending, which is described in the following section. Conceptual blending is a different type of theory than is often used in the mathematics education research literature.

Many frameworks describe student thinking in ways that help us parse various ways a concept may be understood or discussed by students or teachers (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Vinner & Dreyfus, 1989; Zandieh, 2000). Other frameworks allow us to distinguish the structure of one student's argument from another (e.g., Alcock & Simpson, 2004; Harel & Sowder, 1998; Weber & Alcock, 2004). These types of distinctions are important and helpful. Blending theory is different from this set of research in that it is less about categorizing student thinking and more focused on the process by which people create new ideas.

There are other frameworks in mathematics education that describe idea creation. For example the work done using the Action, Process, Object, Schema (APOS) framework (cf., Dubinsky, 1991; Dubinsky & McDonald, 2002) or Sfard's (1991) description of reification. However, these are predominately a description of how students may evolve from understanding a concept in one way to eventually understanding the concept in another way. Blending is different than this in that it is not about changing our view of a particular concept, but rather about bringing together two ways of thinking to create a third.