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A framework for characterizing student understanding of Riemann sums and definite integrals

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ABSTRACT

It has been shown in the literature that students can often evaluate definite integrals by applying the Fundamental Theorem of Calculus or by interpreting an integral as an area under a curve. However, students struggle to solve word problems involving definite integrals, even when the context is quite familiar to the students. This research examines the obstacles calculus students encounter and the ways in which they overcome those obstacles when solving definite integral problems without relating to area under a curve. A framework for characterizing student understanding of Riemann sums and definite integrals is presented and discussed. Results indicate that conceptualizing the product of $f(x)$ and Δx proves to be the most complex part of the problem-solving process, despite the simplicity of the mathematical operations required in this step.

1. Introduction

The concept of the definite integral is imperative for students to understand for several reasons. First, many real world applications involve functions that do not have an antiderivative that can be expressed in terms of elementary functions, such as $f(x) = \exp(-x^2)$, which is based on the normal distribution and used regularly in statistics. Since we cannot apply the Fundamental Theorem of Calculus to functions such as the normal distribution, other methods for evaluating the definite integral, such as Riemann sums, would be needed. While Riemann sums may not be the most efficient method for approximating a definite integral, other methods, such as the trapezoid rule, midpoint rule, or Simpson's rule are based on the structure of the Riemann sum. Thus, an understanding of the structure of a Riemann sum will help students understand these other methods as well. Finally, an understanding of Riemann sums is needed even when a function has an antiderivative that *can* be expressed in terms of elementary functions. Setting up the appropriate definite integral requires the student to know what to integrate, and determining the integrand typically involves an understanding of the structure of the Riemann sum.

This study provides a framework describing various mathematical components of the Riemann sum and definite integral in the context of physical situations (e.g. distance, velocity, force, and energy). The framework (from now on called the Riemann Integral Framework) is based on a decomposition of the Riemann integral into its mathematical components of functions, limits, summations, and products. The research questions this study attempts to answer are:

1. Given the mathematical decomposition of the definite integral, how do students engage with each of the layers within the Riemann Integral Framework?

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2. What obstacles do students encounter within each layer of the Riemann Integral Framework, and how are students able to overcome these obstacles?
3. Are the layers of the Riemann Integral Framework sufficient for analyzing student engagement with word problems involving definite integrals?

2. Literature review

The literature base on student understanding of definite integrals includes work on understanding the symbolic notation (Jones, 2013; Rasslan & Tall, 2002), computing definite integrals (Orton, 1983; Rasslan & Tall, 2002), understanding the Fundamental Theorem of Calculus (Thompson, 1994; Thompson & Silverman, 2008), and representing the definite integral as area under a curve (Orton, 1983; Rasslan & Tall, 2002; Sealey, 2006). Orton's (1983) study was an early investigation of students' ability to carry out various techniques of integration. He found that even the best students from his study had significant difficulty with problems that were "concerned with the understanding of integration as the limit of a sum" (Orton, 1983, p. 9). Orton also found that students were able to evaluate the definite integral, but they often did not know *why* they were doing what they were doing. In other words, students could compute the area under the curve and could apply the Fundamental Theorem of Calculus to evaluate a definite integral, but they did not know why the Fundamental Theorem of Calculus gave the desired result or why area under a curve is represented by the definite integral.

Similarly, Thompson and Silverman (2008) discussed the importance of students being able to conceive of definite integrals as more than just area under a curve, even though textbooks often introduce the concept as area. Specifically, they argued that students must interpret "the quantities being accumulated as being created by accruing incremental bits that are formed multiplicatively" (Thompson & Silverman, 2008, p. 45). Rasslan and Tall (2002) also discussed students' understanding of the definite integral in terms of area under a curve. They found that even students who are above average have difficulty with improper integrals and piecewise defined functions such as the integer-value function.

Along the same lines, an earlier paper by the author (Sealey, 2006) examined a small group of calculus students' abilities to solve definite integral word problems. Data showed that students often "knew" that a word problem involved definite integrals and should be represented by the area under a curve, but some of the students were not able to determine which curve should be graphed to obtain the correct representation. Students who were successful in solving the word problems were able to more carefully examine all of the components that formed the definite integral. Meredith and Marrongelle (2008) found that students were most successful in solving electrostatics problems involving definite integrals when the students could view the definite integral in terms of parts of a whole. They also note that, "understanding the physical situation is necessary, but not sufficient" for being able to solve the problems correctly (p. 573).

On an encouraging note, Thompson (1994) described a situation with a 7th grader named Sue who was asked to approximate the distance traveled if a car accelerated from 50 miles per hour to 60 miles per hour in 1 hour. Sue's response was mathematically equivalent to a Riemann sum approximation, and Sue was able to discuss whether or not her answer was an overestimate or an underestimate. In addition, she was able to describe that she would get a better approximation if she had used smaller intervals. This shows that some of the ideas of Riemann sums and definite integrals are intuitive for students. However, Thompson notes that while rates involving time are more intuitive for students, they are not trivial, and "a further abstraction is required to develop an image of rate" that involves two quantities other than time (Thompson, 1994, p. 232).

3. Preliminary Riemann Integral Framework

In order to gain insight into how students might develop an understanding of the structure of the Riemann integral, I generated a preliminary Riemann Integral Framework based on the mathematical decomposition of the Riemann integral (Sealey, 2006, 2008; Sealey & Oehrtman, 2005, 2007). This framework models Zandieh's (2000) decomposition of the derivative into its mathematical components. The preliminary Riemann Integral Framework decomposes the Riemann integral into four layers corresponding to the mathematical operations involved in computing the Riemann integral in the form $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$, where x_i represents any x-value on the ith subinterval, $\Delta x = (b-a)/n$, a is the left endpoint of the interval, and b is the right endpoint of the interval. The four layers are product, summation, limit, and function. Although Riemann sums may be taken over other partitions, I focus this framework on partitions with uniform subintervals. Analysis of the data guided modification of the framework to reflect the cognitive development of students.

The first layer of the initial Riemann Integral Framework, the Product Layer, is composed of the multiplication of two quantities, $f(x_i)$ and Δx, where $f(x_i)$ may be conceptualized as a rate and Δx as a difference. As an example, if $f(x_i)$ represents the velocity of an object, and Δx represents the time elapsed, then the product $f(x_i)\ \Delta x$ would represent an approximation for the distance traveled on a particular interval. The inclusion of the constants c and $1/c$ is in response to situations in which the given quantities do not directly correspond to the $f(x_i)$ and the Δx. One such example is explained in detail in Section 5.1.4.

The Summation Layer includes the sum from $i = 1$ to $i = n$, giving us the Riemann sum $\sum_{i=1}^{n} f(x_i)\Delta x$. In the previous example, where $f(x_i)$ represents the velocity of an object, and Δx represents the time elapsed, $\sum_{i=1}^{n} f(x_i)\Delta x$ would represent an approximation for the distance traveled for the entire duration, or from a to b, where a and b are the endpoints of the interval. The third layer includes the limit as n approaches infinity of the previous two layers, giving us the Riemann integral.