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Two proving strategies of highly successful mathematics majors

Dov Zazkis[a,*], Keith Weber[b], Juan Pablo Mejía-Ramos[b]

[a] Arizona State University, School of Mathematical and Statistical Science, United States
[b] Rutgers University, Graduate School of Education, United States

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ABSTRACT

We examined the proof-writing behaviors of six highly successful mathematics majors on novel proving tasks in calculus. We found two approaches that these students used to write proofs, which we termed the *targeted strategy* and the *shotgun strategy*. When using a targeted strategy students would develop a strong understanding of the statement they were proving, choose a plan based on this understanding, develop a graphical argument for why the statement is true, and formalize this graphical argument into a proof. When using a *shotgun strategy*, students would begin trying different proof plans immediately after reading the statement and would abandon a plan at the first sign of difficulty. The identification of these two strategies adds to the literature on proving by informing how elements of existing problem-solving models interrelate.

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1. Introduction

One of the central purposes of many university advanced mathematics courses is to increase mathematics majors' abilities to write proofs (Weber, 2001). Unfortunately, even after completing these courses, many research studies indicate that proof writing remains a persistent difficulty for mathematics majors. In many studies, mathematics majors were asked to complete a set of proving tasks; in each case, they collectively proved substantially fewer than half of the statements assigned to them (e.g., Alcock and Weber, 2010; Hart, 1994; Iannone & Inglis, 2010; Ko & Knuth, 2009; Moore, 1994; Weber & Alcock, 2004; Weber, 2001).

There has been substantial research on mathematics majors' difficulties with writing proofs (see Selden & Selden, 2008, for a comprehensive review), which include epistemological, logical, conceptual, and strategic difficulties. By epistemological difficulties we mean that students need to be persuaded by the same types of arguments that mathematicians consider convincing (Harel & Sowder, 1998, 2007). Misalignments between what mathematicians find convincing and what students find convincing may lead to students' difficulties with both production and comprehension of normatively correct proofs. In particular, the goals that students set for themselves during the proving process may be inconsistent with what their course instructor expects of them. By logical difficulties we mean difficulties with formal logic. These include students' understanding of what forms of logical inferences are permissible (e.g., modus tollens) (Weber & Alcock, 2005), students' understandings of multiply qualified statements (Zandieh, Roh, & Knapp, 2014) and how one can logically structure a proof (Savić, 2011; Selden & Selden, 1995). Conceptual difficulties are difficulties with students' understanding of concepts and

* Corresponding author. Tel.: +1 619 817 1898.
E-mail address: zazkis@gmail.com (D. Zazkis).

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definitions (Hart, 1994; Tall & Vinner, 1981). These difficulties may lead students to make false inferences or leave students unable to make a critical deduction that is needed to write a proof. Finally, by strategic difficulties we are referring to issues with students' problem-solving approaches. These include which strategies students use to generate a proof, when and how they choose to abandon particular strategies, and how they decide which strategies are worth pursuing (Anderson, Boyle, & Yost, 1986; Schoenfeld, 1985; VanSpronsen, 2008; Weber, 2001). Although all of these types of difficulties are important, in this work we focus on the last of these categories, students' proving strategies.

One approach to identifying the strategies needed to construct proofs is to carefully study the behavior of those who are proficient at proof writing. There has been limited work in this area. A first body of work has examined the proof-writing behavior of mathematicians. Using an expert-novice research paradigm, Weber (2001) identified proof writing heuristics that mathematicians used in abstract algebra that undergraduates appeared to lack. Lockwood, Ellis, Dogan, Williams, and Knuth (2012) studied the ways in which mathematicians used examples to solve problems and write proofs. Samkoff, Lai, and Weber (2012) explored how mathematicians used diagrams when writing a proof. This focus on mathematicians provides valuable insight into how proofs may be successfully written, but it has an important limitation. Mathematicians' proving strategies might rely on experiences and understandings that most undergraduates may lack. If so, exposing students to these strategies without these corresponding experiences may lead to naïve application of the strategies, which can be counterproductive (cf., Reif, 2008).

A second set of studies has examined the strategies of students who wrote proofs successfully. Gibson (1998) investigated how undergraduates' use of diagrams enabled them to overcome impasses when writing proofs in real analysis. Sandefur, Mason, Stylianides, and Watson (2013) studied how considering examples aided two groups of undergraduates on a specific task in number theory. The goal of this paper is to contribute to this second set of studies. To do so, we studied the proof-writing behavior of highly successful mathematics majors. Specifically, we examined six highly successful mathematics majors who performed well, both in their proof-oriented mathematics courses and in proving tasks that we assigned to them in this study in order to glean their proof strategies.

2. Theoretical framing

In this paper, we view proof writing as a problem-solving task in which the student is given a statement to prove and has the goal of producing a deductive argument that establishes the statement to be proven while conforming with mathematical norms (Furinghetti & Morselli, 2007; Weber, 2005). Rather than problematize the personal goal that students are trying to achieve or the knowledge base they have, we are interested in how the students marshal the resources they have to write proofs. This suggests two strategic difficulties that students may have. The first strategic difficulty involves students' choice of approach. The set of potential approaches to most proving tasks are numerous, but only a fraction of these approaches are likely to be useful. Students may choose a wrong approach and hence not produce a proof (Anderson et al., 1986; Schoenfeld, 1985; Weber, 2001). Hence, how students choose which approach to implement is of particular importance when studying their proof generation strategies.

The second related consideration concerns when students reach an impasse and do not know how to proceed. This can occur when a student reaches a dead end when implementing one approach or simply when the student cannot spontaneously generate any way to approach the task. That is, a student may feel that he\she can no longer take actions that are potentially productive to further progress with their current plan (e.g., Moore, 1994; Schoenfeld, 1980; Weber, 2001). Hence, an important strategic consideration involves examining how students identify when an impasse has been reached and what they do in an effort to overcome it.

2.1. Understanding and planning

The literature suggests several ways of negotiating the interplay between strategy choice and impasses. A common suggestion in the literature is that when students encounter a problem-solving task, they should first spend time working to understand the problem and then carefully choose a plan to solve the problem (Polya, 1945; Schoenfeld, 1985). More recent work has demonstrated a complicated interaction between the stages of understanding, planning, and implementing a plan; indeed, expert problem solvers may form a plan to understand some aspect of a problem that is confusing to them (Carlson & Bloom, 2005). But most models of expert mathematical problem-solving agree that one spends at least some time initially trying to understand a problem before solving it. Students tend not to do this and this is one account given to explain their difficulties with problem-solving (Schoenfeld, 1985).

2.2. Metacognition and monitoring

Schoenfeld (1985) observed that expert problem solvers not only carefully chose their problem-solving plan but they also monitored their progress in applying this plan. In Schoenfeld's studies, mathematicians would repeatedly ask themselves monitoring questions such as, "what is this plan trying to achieve?" and "do I think I can achieve these goals?", using their responses to guide their actions, including switching plans if necessary. In contrast, students tended to be more single-minded, with the result being that they failed to solve a problem since their efforts were entirely comprised of following a plan that was doomed from the start (Schoenfeld, 1987). DeFranco (1996) expanded upon Schoenfeld's results in his