Journal of Mathematical Behavior 39 (2015) 28–50

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Lessons learned from an instructional intervention on proof comprehension

Aron Samkoff, Keith Weber*

10 Seminary Place, New Brunswick, NJ 08901, USA

ARTICLE INFO

Article history:
Available online 12 June 2015

Keywords:
Proof
Proof comprehension
Proof reading strategies

ABSTRACT

In previous research, we proposed a set of proof reading strategies that we hypothesized can help students better understand the proofs that they read. The goal of the present paper is to report lessons that we learned from two instructional interventions in which students were taught to apply these strategies. We found suggestive evidence that implementing these strategies helped students understand the proofs that they read, but also found students' implementation of these strategies to sometimes be problematic. We present instructional modifications, as well as refinements to the strategies themselves, that enabled the students to implement the strategies more effectively.

1. Introduction

Mathematical proof plays a fundamental role in mathematicians' practice. Proof is the primary means by which mathematicians demonstrate that a theorem is true. Further, proofs can serve as bearers of knowledge, both by providing insight for why a theorem is true, and by illustrating problem solving methods that can be used to prove other statements (Rav, 1999; Steiner, 1978). Proofs also play an important role in advanced mathematics courses—i.e., the upper-level proof-oriented university classes for mathematics majors. In these courses, proofs are the dominant form of pedagogical explanation (cf. Lai & Weber, 2014; Lai, Weber, & Mejía-Ramos, 2012), with Mills (2011) estimating that half of the time in mathematics classrooms is spent dealing with proofs. Interviews with mathematics professors indicate that the proofs presented in classrooms are presented so that students might gain techniques and understanding, and not merely conviction that the theorem statement is true (Weber, 2012; Yopp, 2011).

Despite these efforts, studies have suggested that students have significant difficulties with reading[1] the proofs that are presented to them. Mathematics majors are largely unable to determine if a proof is correct (Alcock & Weber, 2005; Inglis & Alcock, 2012; Ko & Knuth, 2013; Selden & Selden, 2003; Weber, 2010) and researchers have remarked that mathematics majors seem to have serious difficulties understanding proofs (Conradie & Frith, 2000; Cowen, 1991). The empirical research on proof reading has largely consisted of either measuring mathematics majors' success at determining if an argument is valid (e.g., Alcock & Weber, 2005; Inglis & Alcock, 2012; Ko & Knuth, 2013; Selden & Selden, 2003; Weber, 2010) or asking students whether an argument is convincing and using these responses to provide insight into students' standards of conviction (e.g.,

* Corresponding author at: Rutgers University, USA. Tel.: +1 848 932 0804.
E-mail addresses: aron.samkoff@rutgers.edu (A. Samkoff), keith.weber@gse.rutgers.edu (K. Weber).

[1] By "reading a proof", we are referring to looking at the proof with the intent of developing meaning from it. By "understanding a proof", we mean developing the facets of the Mejia-Ramos et al. (2012) model which we describe shortly. Reading is a behavioral activity while understanding is a cognitive state. Hence, students will read proofs with the aim of understanding them, and they may or may not be successful at achieving this goal. As we argue in this paper, the research literature suggests that mathematics majors frequently do not understand the proofs that they read.

http://dx.doi.org/10.1016/j.jmathb.2015.05.002
0732-3123/

Healy & Hoyles, 2000; Martin & Harel, 1989; Segal, 2000; Weber, 2010). Mejia-Ramos and Inglis (2009) argued that there has been comparatively little research on students' *comprehension* of proofs. Because undergraduates encounter many proofs in their advanced mathematics courses that are supposed to increase their mathematical understanding, there is an urgent need for research on ways in which students can be successfully taught to improve their reading comprehension. Our goal in this paper is to contribute toward this gap in the literature.

In our previous work, we identified proof reading strategies (Weber, in press; Weber & Samkoff, 2011) that we hypothesized had the potential to improve mathematics majors' comprehension of the proofs that they read. We used this research as a starting point for an instructional experiment where we tried to teach students these strategies using a modeling–scaffolding–fading approach (Brown, Collins, & Newman, 1989). By analyzing this intervention, we provide suggestive confirmatory evidence that some of these strategies can improve some students' proof comprehension. We also illustrate that sometimes students do not benefit from using these strategies, largely because students could not implement these strategies correctly. In these cases, we present an analysis of why students failed to correctly implement the strategies and suggest modifications to how the strategies were introduced to students or to the strategies themselves that might be beneficial for future rounds of the study.

2. Theoretical perspective

In this paper, we incorporate the model of Mejia-Ramos, Fuller, Weber, Rhoads, and Samkoff (2012) for characterizing and assessing students' understanding of a proof in advanced mathematics. In this model, there are seven dimensions to understanding a proof: three "local" and four "holistic" dimensions. The local dimensions deal with understanding that can be gleaned from carefully reading a small number of statements in the proof, and consist of the following:

(1) Meaning of terms and statements: Understanding the meaning of terms and individual statements of the proof. This includes stating the definitions of terms used in the theorem statement and proof and identifying trivial implications of a given statement.
(2) Justification of claims: Understanding why each claim made in the proof follows from previous ones, and being able to identify claims that follow from a given statement later in the proof.
(3) Logical status of statements and proof framework: Understanding the logical relation between the assumptions and conclusions in a proof, identifying the proof technique being used, and conceptualizing the proof in terms of its proof framework (cf. Selden & Selden, 1995).

"Holistic" ways to understand a proof concern synthesizing the entire proof or entire parts of the proof as a coherent whole, and include:

(1) Identifying the modular structure: Understanding how a proof can be broken into mathematically independent parts or sub-proofs, and how these parts logically relate to one another.
(2) Illustrating with examples: Understanding how a sequence of inferences can be applied to verify that a general theorem is true for a specific example.
(3) Summarizing via high-level ideas: Understanding the overarching logical structure of the proof and being able to summarize a proof in terms of these ideas.
(4) Transferring the general ideas or methods to another context: Being able to use the ideas or methods in the proof to establish a different theorem.

This model was used to specify the learning goals of the instructional experiment; we wanted students to improve their understanding of proofs as judged by this model. The model was also used to generate assessment questions that were asked to students after each proof, which was done to provide students with feedback on their own understanding. This is further described in the 'Methods' section of the paper.

3. Strategies for understanding proofs

Weber (in press) observed two pairs of successful mathematics majors trying to understand six mathematical proofs and identified six potentially useful strategies that these students used along with a theoretical rationale for why these strategies might improve comprehension. Weber (in press) then surveyed 83 mathematicians about whether they desired that mathematics majors use these strategies. For five of these strategies, most mathematicians claimed that they desired their students use these strategies. When Weber and Mejia-Ramos (2013a) surveyed 175 mathematics majors about whether they used these strategies when reading proofs, there was no strategy that the majority of mathematics majors claimed that they used. Consequently, Weber and Mejia-Ramos (2013a) conjectured that mathematics majors' understanding could improve if they could be taught to apply these strategies. In the current paper, we explore this conjecture.

In addition to these five strategies, we also included some strategies that were suggested in the literature on proof. Below, we present each strategy that we used, as well as a description of what the successful implementation of each