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The incubation effect: How mathematicians recover from proving impasses

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ABSTRACT

The literature on mathematicians' actions during proving has, thus far, been primarily anecdotal. This paper reports the *observed* actions of nine mathematicians, six of whom came to an impasse while constructing proofs alone on an unfamiliar topic, from a set of notes, and with unlimited time. The existence of impasses and the actions participants took to recover from them were either directly observed from the real-time data collected (using an innovative technique) or obtained from exit interviews or focus groups. Certain times could be considered a period of *incubation*, which psychologists have defined as a "temporary shift away from an unsolved problem that allows a solution to emerge seemingly as if from no additional effort" (Sio & Ormerod, 2009, p. 94). These actions to overcome impasses, while naturally part of mathematicians' proving processes, could be discussed with students in a classroom setting to help alleviate difficulties in proving.

The proving process has been examined at the university level from various viewpoints including: students' difficulties with the overall process (Moore, 1994; Weber & Alcock, 2004), students' difficulties with validations of proofs (Selden & Selden, 2003), students' difficulties with comprehension of proofs (Conradie & Frith, 2000; Mejia-Ramos et al., 2010), as well as by the categorization of students' proof schemes (Harel & Sowder, 1998), that is, by the ways university mathematics students decide what is convincing and persuading. This study takes up another important aspect of the proving process that has as yet been discussed mainly anecdotally (Hadamard, 1945; Liljedahl, 2004; Poincaré, 1946), namely, how mathematicians respond to, and often overcome "getting stuck."

1. Background literature

1.1. Impasses, incubation, and insight in psychology and neuroscience literature

In examining mathematicians' proving practices, the study reported here focused on impasses and how the mathematicians overcame those impasses, including incubation and the resulting insights. The motivation for this examination is highlighted by Sio and Ormerod (2009), who stated that, "understanding the role of incubation periods may also allow us to make use of them effectively to promote creativity in areas such as individual problem solving, classroom learning, and work environments" (p. 94). Impasses, incubation, and insight have been examined in the psychology and mathematics education literatures, mainly in analyzing problem solving, but there has been little research on them during proving. A

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brief discussion of these literatures provides background for the use of the terms impasse and incubation in examining and analyzing proof construction.

In the psychology literature, an impasse is defined as a state of mind where problem solving attempts cease and the impression arises that the problem is unsolvable (Glatzeder, Goel, & von Müller, 2010, p. 17). Problem solvers in the psychology literature sometimes recovered from impasses through *incubation*. Incubation, according to Wallas (1926), is the process by which the mind goes about solving a problem, subconsciously and automatically. It is the second of Wallas' four stages of creativity, which are:

- preparation (thoroughly understanding the problem),
- incubation (when the mind goes about solving a problem subconsciously and automatically),
- illumination (internally generating an idea after the incubation process, sometimes known as the Aha! experience), and
- verification (determining whether that idea is correct).

Incubation has also been described as "a gradual and continuous unconscious process . . . during a break in the attentive activity toward a problem" (Segal, 2004, p. 141). Neuroscientists have researched incubation using fMRI technology (Binder et al., 1999; De Luca, Beckmann, De Stefano, Matthews, & Smith, 2006). They have found that during incubation "the brain contains highly organized, spontaneous patterns of functional activity at rest" (Buckner & Vincent, 2007, p. 2). In addition, Smith and Blankenship (1991) stated that "the time in which the unsolved problem has been put aside refers to the *incubation time*; if insight [illumination] occurs during this time, the result is referred to as an *incubation effect*" (p. 61). It has been conjectured that this effect happens best when one takes a break from creative work (Krashen, 2001).

There seem to be at least two incubation techniques that have yielded positive effects during problem solving: deliberately taking breaks and using "low-demand" tasks. Deliberate incubation has been shown to result in a greater incubation effect than merely being interrupted during the problem-solving process: "Individuals who took breaks at their own discretion (a) solved more problems and (b) reached fewer impasses than interrupted individuals" (Beeftink, van Eerde, & Rutte, 2008, p. 362). Scientists have studied which tasks can be done during those deliberate breaks. In their meta-analysis of 29 articles covering 117 separate psychology experiments dealing with incubation, Sio and Ormerod (2009) stated that "low-demand tasks[1]" done during an incubation period yielded positive incubation effects compared to "high-demand tasks[2]": "There remains a possibility, of course, that a sufficiently light load might allow additional covert problem solving compared with a heavier task load" (p. 107).

1.2. Impasses and incubation in mathematics and mathematics education

There is an extensive problem-solving literature (e.g., Carlson & Bloom, 2005; Schoenfeld, 1992) but for this study, the concentration is on how provers handle impasses, which includes experiencing periods of incubation. To date, research on incubation during problem solving in the mathematics education literature has been sparse and primarily anecdotal. Perhaps this is because creativity, which includes incubation, has rarely been captured in mathematics education research: "[S]tudying a mathematician's or student's creativity is a very difficult enterprise because most traditional operationalized instruments fail to capture extra cognitive traits, such as beliefs, esthetics, intuitions, intellectual values, self-imposed subjective norms, spontaneity, perseverance standards, and chance" (Freiman & Sriraman, 2007, p. 23). Some instruments that have been used to capture creativity in mathematics education include video interviews, written work, or problem/proving sessions in front of a camera (Garii, 2002).

Mathematicians have sometimes acknowledged that preparation is a requirement for creativity. Poincaré (1958) described explicitly a time during which he experienced an insight after an incubation period. He believed that the "preparation stage" along with incorrect attempts on proofs is more useful than one usually thinks, believing it sets the unconscious mind at work. Observation of the unconscious mind is beyond the scope of this study, yet Poincaré (1958) and Hadamard (1945) both devoted many thoughts and conjectures to this.

The mathematician Mordell (1959) suggested that mathematicians need to be motivated by, and immersed in, a problem for creativity, including incubation, to occur. Byers (2007), another mathematician, described stages similar to those of Wallas (1926) and Poincaré (1958), but seemed to focus as much on finding as on proving theorems. He stated that:

> The mathematician's work can be broken down into various stages. The first involves spade work: collecting data and observations, performing calculations, or otherwise familiarizing oneself with a certain body of mathematical phenomena. Then there are the first inklings that there exists in this situation a pattern or regularity—something that is going on. This is followed by the hard work of bringing the embryo into fruition. Then, finally, when the idea has appeared, there is the stage of verification or proof. (pp. 196–197)

[1] Examples of low-demand tasks demonstrated in incubation experiments (Sio & Ormerod, 2009) include listening to music or memorization of a passage.
[2] An example of a high-demand task demonstrated in incubation experiments (Sio & Ormerod, 2009) included the "farm problem," which asks to "divide an L-shaped farm into four parts that have the same shape and size."