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Teaching eigenvalues and eigenvectors using models and APOS Theory

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ABSTRACT

In this article we share the results of an investigation of a classroom experience in which eigenvalues, eigenvectors, and eigenspaces were taught using a modeling problem and activities based on APOS (Action, Process, Object, Schema) Theory.

We show how a sample of 3 students were able to construct an object conception of these difficult concepts in one semester course—something that existing literature had proven to be almost impossible to achieve. Using one team as a case study we describe the work done by a group of 30 students to show how eigenvectors and eigenvalues emerged in a group discussion. Furthermore, we present evidence on how, at least three students, were able to construct an object conception, demonstrating a deep understanding of these concepts. Finally, we validate the designed genetic decomposition. In summary, the results demonstrate the approach to be promising in the learning of eigenvalues and eigenvectors.

1. Introduction

Linear Algebra has become a mandatory course in many undergraduate degrees since it is widely recognized to have many important applications in different disciplines. Larson, Rasmussen, Zandieh, Smith, and Nelipovich (2007), Sierpinska (2000), Possani, Trigueros, Preciado, and Lozano (2010), among others, have found that both teachers and students consider this subject difficult. They have carried out research to study the obstacles that students face when learning concepts of Linear Algebra, how students learn its different concepts, and how they can be helped in the learning process.

Different researchers have developed projects to show that it is possible to teach different mathematical concepts using problems in context and modeling techniques (for example, Possani et al., 2010; English, Lesh, & Fennewald, 2008; Schorr & Lesh, 2003; Vargas, 2013; Fonseca, Pereira, & Casas, 2011; Aguirre, Elguero, & Rosso, 2006). Some researchers agree on the usefulness of modeling in mathematical teaching. Their investigations show that students

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struggle to solve problems and to apply concepts to other disciplines, when they have studied them only from a theoretical point of view (Lesh & Doerr, 2003; Kwon, 2002; Blum, 2009; Haines, 2009; Bas, Cetinkaya, & Kursat, 2009). According to these studies, modeling motivates students to get involved in mathematic courses and to raise their interest in learning new concepts. They agree as well that modeling is useful in identifying the difficulties students face while working with problems and that it helps students comprehend specific mathematical concepts. Some of them believe that modeling can also facilitate learning and concept construction, allowing students to approach problems with a broader range of strategies. Furthermore, several investigations (Possani et al., 2010; Perdomo, 2011; Trigueros, 2009; Rasmussen, Zandieh, King, & Teppo, 2005; Camacho, Perdomo, & Santos, 2012) have shown that students with average ability can develop powerful models to describe complex systems that only depend on a new application of relatively elemental mathematical concepts. Our research also shows that modeling can be used to introduce new concepts.

To contribute to the understanding of teaching and learning processes in Linear Algebra, we have researched modeling as an approach to introduce abstract concepts, and analyzed if using this methodology, together with a theory of mathematics education, contributes to students' learning of the abstract concepts of this discipline. In this paper we focus on the application of Models and Modeling and APOS (Action, Process, Object, Schema) Theory to introduce the eigenvalue, eigenvector, and eigenspace concepts in a Linear Algebra course. We chose APOS Theory as it focuses on how students construct different mathematical concepts and how to use this information to propose pedagogical actions that can stimulate the learning process. APOS Theory provides well defined constructs to describe and compare students' development while they learn different concepts. It enables researchers to compare differences among students who work using memorized facts (Action conception) and those who are able to apply the concepts in the solution of new problems and even determine their properties and relate it to other concepts (Object conception). Teaching and research results have shown that developing an object conception for a concept is difficult and takes a long time (Asiala, Brown, Kleiman, & Mathews, 1998; Arnon et al., 2014; Clark, Kraut, Mathews, & Wimbish, 2007; Trigueros & Martínez-Planell, 2010). However, it has also been demonstrated that teaching approaches following APOS Theory have been successful in helping students to attain a richer understanding of several mathematical concepts (Dubinsky & McDonald, 2001; Weller et al., 2003).

The research questions considered in this study are: Which strategies do students use when facing a modeling problem whose solution requires eigenvalues, eigenvectors, and eigenspaces? Can students construct an object conception of eigenvalue, eigenvector, and eigenspace when these concepts are taught using a didactical design based on APOS Theory and the use of models?

In what follows, we first present a summary of the results from different studies related to the present research. We then introduce the theoretical framework and describe the methodology we applied in this study. We designed, using APOS methodology, a genetic decomposition where constructions needed to learn a mathematical concept are described in detail. We contrast and validate, or refine, the genetic decomposition using experimental results. We include in the methodology section the genetic decomposition designed for this study. We present and discuss in detail the results of our findings and our proposed answers to the research questions above.

2. Background

There is little research in how the concepts of eigenvalues, eigenvectors, and eigenspaces can be learned and what are the main difficulties encountered by students. In this section, we summarize the results of investigations on eigen theory.

Stewart and Thomas (2007), Thomas and Stewart (2011) found students' struggled with the manipulation of the definition of eigenvalues and eigenvectors. Using an embodied theoretical position to analyze the way in which students understand these concepts, they found that eigenvalues and eigenvectors are usually introduced in class through formal mathematical definitions without any motivation and that this leads students into a world of algebraic and matrix manipulations, such as transforming $\boldsymbol{A}\bar{\boldsymbol{v}} = \boldsymbol{\lambda}\bar{\boldsymbol{v}}$ into $(\boldsymbol{A} - \boldsymbol{\lambda}\boldsymbol{I})\bar{\boldsymbol{v}} = \bar{\boldsymbol{0}}$, without a clear understanding of the concept. Their results showed that algebraic manipulation can become an obstacle in the understanding of the concepts of eigenvalue and eigenvector. On one hand the equation $\boldsymbol{A}\bar{\boldsymbol{v}} = \boldsymbol{\lambda}\bar{\boldsymbol{v}}$ is composed by $\boldsymbol{A}\bar{\boldsymbol{v}}$ and $\boldsymbol{\lambda}\bar{\boldsymbol{v}}$ which are two different processes: on the left there is a matrix times a vector while on the right there is a vector times a scalar, and students are not aware that both products result in vectors that may be compared in terms of their equality. On the other hand, when transforming the definition, students show a tendency not to include the identity matrix needed in $(\boldsymbol{A} - \boldsymbol{\lambda}\boldsymbol{I})\bar{\boldsymbol{v}} = \bar{\boldsymbol{0}}$ and they don't understand its meaning as identity. This fact is usually not explained by most textbooks and even when it is explained students have problems to understand why $\boldsymbol{\lambda}$ is substituted by $\lambda\boldsymbol{I}$. These authors concluded that it is important to emphasize the role of the identity matrix in general when doing matrix manipulations. They also showed the value of explicitly presenting the actions and procedures of the symbolic world to find them and the need to relate these actions and procedures with conceptual ideas such as the number of possible eigenvectors.