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Identifying a framework for graphing formulas from expert strategies

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ABSTRACT

It is still largely unknown what are effective and efficient strategies for graphing formulas with paper and pencil without the help of graphing tools. We here propose a two-dimensional framework to describe the various strategies for graphing formulas with recognition and heuristics as dimensions. Five experts and three secondary-school math teachers were asked to solve two complex graphing tasks. The results show that the framework can be used to describe formula graphing strategies, and allows for differentiation between individuals. Experts used various strategies when graphing formulas: some focused on their repertoire of formulas they can instantly visualize by graphs; others relied on strong heuristics, such as qualitative reasoning. Our exploratory study is a first step towards further research in this area, with the ultimate aim of improving students' skills in reading and graphing formulas.

1. Introduction

Students often have difficulties with algebra, in particular giving meaning to and grasping the structure of algebraic formulas, and manipulating them (Chazan & Yerushalmy, 2003; Drijvers, Goddijn, & Kindt, 2010; Kieran, 2006; Sfard & Linchevski, 1994). Functions can be represented in several forms, such as algebraic formulas and graphs; the latter are more accessible for students than the former (Janvier, 1987; Leinhardt, Zaslavsky, & Stein, 1990; Moschkovich, Schoenfeld, & Arcavi, 1993).

A graphical representation gives information on covariation, that is, how the *y*-coordinate (the dependent variable) changes as a result of changes of the *x*-coordinate (the independent variable) (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). A graph shows possible symmetry, intervals of increase or decrease, extreme values, and infinity behavior. In this way, it visualizes the "story" of an algebraic formula. Graphs may help learners to give meaning to algebraic formulas and so make learning algebra easier for them (Eisenberg & Dreyfus, 1994; Kilpatrick & Izsak, 2008; NCTM, 2000; Philipp, Martin, & Richgels, 1993; Yerushalmy & Gafni, 1992).

Graphs are also considered important in problem solving (Polya, 1945; Stylianou & Silver, 2004). In his list of heuristics, Polya (1945) mentions drawing a picture or diagram as one of the first options. Creating and using multiple representations,

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and switching between them, are important tools in problem solving (Janvier, 1987; NCTM, 2000). Stylianou and Silver (2004) and Stylianou (2002, 2010) found how graphs are used to understand the problem situation, to record information, to explore, and to monitor and evaluate results.

For learning about functions, graphing tools such as graphic calculators are recommended (Drijvers & Doorman, 1996; Drijvers, 2002; Hennessy, Fung, & Scanlon, 2001; Kieran & Drijvers, 2006; Philipp et al., 1993; Schwartz & Yerushalmy, 1992; Yerushalmy & Gafni, 1992). With these tools, graphing formulas seems easy. In the past, constructing a graph was itself a goal or the graph itself an end product. To produce one, many algebraic skills (determining domain, zeroes, derivative, etc.) were employed, along with standard methods requiring multiple algebraic manipulations, which were not straightforward for all learners.

Graphing tools now make it possible to study problems that in the past could not be solved or could be solved only with difficulty. In order to use these tools adequately, however, one must know what aspects of graphs to look for (Philipp et al., 1993). According to Stylianou and Silver (2004) novices experience difficulties in the visual explorations of the graphs they have constructed. They concluded that such explorations are restricted to familiar functions. So, in order to make effective and efficient use of technology, learners should know about graphs representing basic functions, and also should have learned to reason about such graphs (Drijvers, 2002; Eisenberg & Dreyfus, 1994; Stylianou & Silver, 2004).

Learners who do graphing with pen and paper may establish the connection between the algebraic and the graphical representations of a function more effectively than learners who only perform computer graphing (Goldenberg, 1988). In this article, graphing to produce a sketch of a graph with its main characteristics without technological help will be called *graphing formulas*.

Despite earlier research on how to learn and how to teach functions, it is still largely unknown what knowledge and skills are necessary to graph formulas effectively and efficiently. In order to learn more about these, we have identified expert strategies in our research. Experts are expected to know and use more effective and efficient strategies than novices (Chi, 2006, 2011). Hence, the focus of this article will be on determining a suitable framework for formula graphing strategies. With the help of this knowledge base, a professional development trajectory for teachers and teaching material for students may eventually be developed.

2. Theory

2.1. Aspects of graphing formulas

Functions are at the core of math education. There are several reasons for students' difficulties with the concept. Functions, like other mathematical concepts, are not directly accessible as physical objects. Access to mathematical concepts can only be gained through representations. To understand mathematical concepts one needs to relate elements of different representations (Janvier, 1987; Kaput, 1998). For functions, these representations are algebraic formulas, graphs, tables, and contexts (Janvier, 1987). These representations have to be combined in order to produce a rich concept image of the function (Thomas, Wilson, Corballis, Lim, & Yoon, 2010; Tall & Vinner, 1981).

The ability to represent concepts, to establish meaningful links between and within representations, and to translate from one representation of a concept to another is at the core of doing and understanding mathematics. Different concepts have been used to refer to this ability: representational flexibility (Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009), representational fluency (Lesh, 1999), representational versatility (Thomas & Hong, 2001). 'Representational versatility' has been defined as the ability to work seamlessly within and between representations and to engage in procedural and conceptual interactions with representations (Thomas et al., 2010). Our research deals with translations between algebraic formulas and graphs, demonstrating representational versatility.

Much research has been done on algebraic and graphical representations and their relations. Students are often found to have difficulties with reading algebraic formulas and the so-called process-object character of a function. For graphing formulas, it is necessary that one can "read" algebraic formulas and deal with the process-object character of a function. These two issues are discussed in the next sections.

2.1.1. Reading algebraic formulas

There are different ways to create meaning for algebraic formulas: from the problem context, from the algebraic structure of the formula, and from its various representations (Kieran, 2006). In order to read an algebraic formula, one has to grasp its structure (Sfard & Linchevski, 1994). In the literature this is called 'symbol sense' (Arcavi, 1994). Symbol sense has several aspects, such as the ability to read through algebraic expressions, to see the expression as a whole rather than a concatenation of letters, and to recognize its global characteristics (Arcavi, 1994). Symbol sense enables people to scan an algebraic expression so as to make rough estimates of the patterns that would emerge in numeric or graphical representations (Arcavi, 1994).

A procedure for analyzing the syntactic structure of an expression was formulated by Ernest (1990). A syntactical tree is constructed via an iterative procedure in which the main operator of the expression is identified. The procedure continues until all subexpressions have been given meaning. The decomposition of algebraic expressions into meaningful parts (building blocks) can be considered a heuristic for reading formulas.