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The duality between ways of thinking and ways of understanding: Implications for learning trajectories in mathematics education

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ABSTRACT

The purpose of this paper is to argue that attention to students' ways of thinking should complement a focus on students' understanding of specific mathematical content, and that attention to these issues can be leveraged to model the development of mathematical knowledge over time using learning trajectories. To illustrate the importance of ways of thinking, we draw on Harel's (2008a, 2008b) description of mathematical knowledge as comprised of *ways of thinking* and *ways of understanding* to characterize students' thinking about mathematics in two case studies. We use these case studies to illustrate the explanatory and descriptive power that attention to the duality of ways of understanding and ways of thinking provides, and we propose suggestions for constructing learning trajectories in mathematics education research.

1. Introduction

In recent years, research that looks deeply at student learning has flourished, and research that focuses on the development of learning trajectories has received particular attention. Much of this research has taken substantial strides in articulating students' mathematical knowledge about particular content areas such as fractions (Simon & Tzur, 2004), partitioning and splitting (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009), and length measurement (Barrett et al., 2012; Sarama, Clements, Barrett, Van Dine, & McDonel, 2011; Szilagyi, Clements, & Sarama, 2013), allowing researchers and teachers to gain much insight about students' thinking about such topics. Our reading of the work cited above, as well as our own efforts to document student thinking, suggest an important distinction between two aspects of mathematical knowledge – first, knowledge about mathematical content (i.e., knowledge of a particular proof) in particular, and second, broader aspects about that content knowledge (i.e., knowledge of what constitutes a proof) that supersede, but also interact with, knowledge about particular mathematical concepts.

We contend that both of these dimensions of mathematical knowledge should be incorporated into research involving learning, and, more specifically, learning trajectories, and we also recognize that some researchers have begun inroads into this issue (Clements & Sarama, 2009; Sarama & Clements, 2009). However, we think that incorporating both dimensions necessarily entails considering them as reflexive, and that the reflexivity between them provides a means to explain conceptual change in learning trajectories. Thus, we see this paper's contribution not only as a call to continue the focus mathematical

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content knowledge *and* characteristics of that content knowledge, but also as a means to think about how those two are related and influence each other. More specifically, we propose that Harel's (2008a, 2008b, 2008c) Duality principle is a useful theoretical lens through which to consider this relationship. Moreover, we want to emphasize that while our approach in using the duality principle to think about learning trajectories is new, the constructs on which we build (LTs and the duality principle) come from existing bodies of literature.

2. Motivation for duality in learning trajectories

The distinction between content knowledge and broader aspects about that content knowledge we drew in the previous section is not necessarily novel, though it has long been intuitive, not yet defined cognitively and epistemologically. For instance, the CCSSM distinguishes between standards of mathematical content and standards of mathematical practice, which are essentially characteristics of using mathematics (National Governors Association Center for Best Practices, 2010). Cuoco, Goldenberg, and Mark (1996) talk about "habits of mind" related to doing mathematics, and Harel's (2008a, 2008b, 2008c) ways of thinking and ways of understanding represent an important distinction as well.

There is some evidence of this distinction being operationalized by those focusing on learning trajectories. As a first example, Ellis, Ozgur, Kulow, Dogan, and Williams (2013) describe a learning trajectory for exponential growth that focuses on students' notions of function. In this learning trajectory, Ellis and her colleagues attend to the learning of particular mathematical content, such as the fact that rate of change of an exponential function is proportional to the amount of change in the function's argument and they suggest sequences of tasks that help students understand such content knowledge. They complement the focus on content by proposing two fundamental views of exponential growth, which they identify as a correspondence view and a covariation view in their learning trajectory. Ellis and colleagues' work highlights each view as being a perspective that a student may have about exponential growth. For instance, those students with what they call a *correspondence view* might believe that the growth factor has a greater effect on growth than the initial amount as x increases to sufficiently large values, whereas students with the *covariation view* might think that y increases multiplicatively by the growth factor for any unit change in x. Each of these views represents not only a particular aspect of content knowledge, but also suggests broad characteristics of how students approach exponential growth. At the same time the development of the covariation and correspondence views seems qualitatively different from the development of content knowledge such as the meaning of the base in an exponential function. Indeed, they seem to be broader aspects about content knowledge that appear repeatedly in the learning of exponential growth. In some sense, they appear to affect the ways in which particular content can be understood. Ellis and her colleagues' learning trajectory is an example of how students' knowledge might develop over time, where that development entails two aspects of learning: the learning of content knowledge (i.e. y increases multiplicatively by the growth factor for a unit change in x) and characteristics of that content knowledge (i.e. a covariation or correspondence view).

Our argument that mathematical knowledge may consist of both the knowledge of content and broader characteristics of that content knowledge is not new, but consideration of how explicit attention to this distinction might affect the development of learning trajectories has not yet been addressed. The importance of this distinction has been noted elsewhere. For example, Empson (2011) characterized the challenges of creating a trajectory about mathematical practices.

Most, if not all, current characterizations of learning trajectories do not address the practices that engender the development of concepts – although it's worth thinking about alternative ways to characterize curriculum standards and learning trajectories that draw teachers' attention to specific aspects of students' mathematical practices as well as the content that might be the aim of that practice (Empson, 2011, p. 573).

These examples illustrate two potential directions in which existing research on learning trajectories might be expanded. First, mathematical learning represented in trajectories often includes content and broader aspects about that content, though often the focus on broader aspects is implicit and could be made more explicit. Second, trajectories consist of conceptual "levels" but they tend not to address the learning that takes place between those levels or what mechanism(s) drives that learning. We believe we can shed light on both of these issues with (a) more explicit attention to how students come to develop knowledge about the characteristics of mathematical ideas (i.e. what constitutes a proof, a covariation view of exponential growth), and (b) focus on knowledge about the broader aspects of content knowledge in a way that considers its reflexive relationship with content knowledge.

To accomplish these aims, we draw on Harel's (2008a, 2008b, 2008c) description of mathematical knowledge as represented by the dual constructs of *ways of thinking* and *ways of understanding* (defined momentarily), each of which influences the other, to characterize students and experts' thinking about mathematics. By using Harel's framework, which emphasizes the reflexive relationship and interaction between these two aspects of mathematical knowledge, we argue that this perspective carries particular benefit for researchers designing and developing learning trajectories. Our reason for focusing on Harel's characterization of this distinction in mathematical knowledge we made earlier is because we feel there is power in the duality he describes. The duality captured in the relationship between ways of thinking and ways of understanding helps to uncover new insights for researchers about students' conceptual change, and these insights can lead to learning trajectories explain the nuances of how students move between various conceptual levels.

In Section 3, we characterize Harel's (2008a, 2008b) Duality-Necessity-Repeated Reasoning (DNR) framework and his definition of mathematical knowledge. His work provides the motivation for the focus on ways of thinking, and we frame our proposed contribution to learning trajectories in terms of his ideas. Then, in Section 4, we describe learning trajectories