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Visualizing number sequences: Secondary preservice mathematics teachers' constructions of figurate numbers using magnetic color cubes

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ABSTRACT

This study is about preservice secondary mathematics teachers' visualization of summation formulas modeled by magnetic color cubes representations. The theoretical framework for this research draws from studies on quantitative reasoning (Smith & Thompson, 2008; Thompson, 1995) and quantitative transformations (Schwartz, 1988). Data consist of videotaped qualitative interviews during which preservice mathematics teachers were asked to construct growing rectangles representing summation formulas. Data analysis is based on analytic induction and constant comparison methodology. Preservice teachers provided a diversity of additive and multiplicative visualizations. Results indicate that quantitative reasoning and mapping structures are fundamental constructs in establishing additive and multiplicative visualizations, hence constructing summation formulas meaningfully. Preservice teachers often had difficulties in explaining the relationships between the same-valued linear and areal quantities. They also established the rectangle condition as the essence of multiplicative visualization.

1. Introduction

> When the successive odd numbers are set forth indefinitely, beginning with 1, observe this: The first one makes the potential cube: the next two, added together, the second; the next three, the third; the four next following, the fourth; the succeeding five, the fifth; the next six, the sixth; and so on. (Nicomachus of Gerasa, Book II, chap. XX).

Figurate numbers, defined as "numbers whose geometric representation took on the form of the various polygons," are numbers that can be expressed as a sum or as a product (NCTM, 1989, p. 53). An identity of the form "sum = product" is often named as a summation formula in the mathematics curricula. Representation of irreducible quantities as well as the larger quantities generated by irreducible quantities evokes the process of unitizing (Behr, Harel, Post, & Lesh, 1994). Summation identities modeled by one-inch color cubes require more sophisticated strategies (additive and multiplicative) at the same time. Conceptual Field Theory (Vergnaud, 1983, 1988, 1994) aims to present the complexity inherent in the nature of "simple" tasks on additive and multiplicative reasoning. Research indicates that the Multiplicative Conceptual Field is very complex and has many concepts of mathematics in its structure, other than multiplication itself (Behr, Harel, Post, &

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Lesh, 1992; Harel & Behr, 1989; Harel, Behr, Post, & Lesh, 1992). “Additive reasoning develops quite naturally and intuitively through encounters with many situations that are primarily additive in nature” (Sowder et al., 1998, p. 128). Building up multiplicative reasoning skills, on the other hand, is not obvious; schooling and teacher guidance are essential to acquire a profound understanding and familiarization with multiplicative situations (Hiebert & Behr, 1988; Resnick & Singer, 1993).

The present study is informed by research studies that investigated students’ content knowledge development in various strands of mathematics; in particular, algebra and algebraic thinking (Chazan & Yerushalmy, 2003; Chazan, 1996; Kirshner, 1989; Leinhardt, Zaslavsky, & Stein, 1990; Sfard, 1991, 1995; Usiskin, 1988; Vergnaud, 1989; Vinner & Dreyfus, 1989), problem solving and modeling (diSessa & Sherin, 2000; Lesh & Harel, 2003; Polya, 1957; Schoenfeld, 1992; Stein, Boaler, & Silver, 2003), learning and teaching of proof (Harel, 1999; Koedinger, 1998; Morris, 2002; Weber, 2001), geometry and spatial reasoning (Burger & Culpepper, 1993; Chazan, 1993; Clements & Battista, 1992; Jones, 2000; Lampert, 1993; Usiskin, 1987; Yerushalmy, 1993), concepts of sequences and series (Alcock & Simpson, 2004; Przenioslo, 2005; Sierpinska, 1990), and students’ understandings and interpretations of representations and modeling in mathematics (Arcavi, 2003; Izsák, 2003; Kaput, 1989; Meira, 1998).

2. Theoretical perspectives

In his 1983 article, Vergnaud defines the notion *conceptual field* as a “set of problems and situations for the treatment of which concepts, procedures, and representations of different but narrowly interconnected types are necessary” (p. 128). In particular, he views the multiplicative structures, a conceptual field of multiplicative type, as a system of different but interrelated concepts, operations, and problems such as multiplication, division, fractions, ratios, similarity. Although multiplicative structures can be modeled by additive structures, they have their own characteristics inherent in their nature, which cannot be explained solely by referring to additive aspects. Behr et al. (1994) developed two representational systems in an attempt to transcribe students’ additive and multiplicative structures in which the notion of units of a quantity plays the main role.

Quantitative reasoning is a central issue in mathematics and science (Smith & Thompson, 2008; Thompson, 1988, 1989, 1993, 1994, 1995). Schwartz (1988), Shalin (1987), and Nesher (1988) view quantities as some sort of mathematical objects as ordered pairs of the form (number, measurement unit) whereas Thompson finds this characterization inconvenient, claiming it “confounds notions of number and quantity” (1994, p. 197). According to Thompson, “A quantity is not the same as a number. A person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the *possibility* of measuring it” (1993, p. 197). He schematizes a quantity as composed of “an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality” (1994, p. 184). Naming, quantifying, attributing units for, and reasoning quantitatively about things are of paramount importance. “Quantitative reasoning is not reasoning about numbers; it is about reasoning about objects and their measurements and relationships among quantities” (Thompson, 1995, p. 204). Reasoning quantitatively about objects, brings with itself the notion of “quantitative operations” by which we make sense of these objects and reason about relationships among them. Quantitative operations are not necessarily the same as the well — known numerical operations of addition, multiplication, subtraction, and division. Thompson views quantitative operations as “conceptual operations one uses to *imagine* a situation and to *reason about* a situation — often independently of any numerical calculations” (1995, p. 207).

Quantitative operations can also be classified as *referent preserving compositions* and *referent transforming compositions* (Schwartz, 1988, p. 41). Addition and subtraction operations are referent-preserving compositions because they do not change, rather preserve the referents of the quantities on which they act. Through a *referent preserving composition*, both the referent and the measurement unit remain unchanged, and we reside in the same measure space. Multiplication and division operations, on the other hand, are referent transforming compositions because they change the referents of the quantities on which they act. Through a *referent transforming composition*, the referent, the measurement unit, and the measure space change. Product quantities such as “2 blouses $\times$ 3 skirts,” “5 $N \times 7\, m$,” “2 in $\times$ 4 in,” involve measurement units of product type “blouse $\times$ skirt,” “$N \times m$,” “inch $\times$ inch,” that are not simply conceived by students as repeated addition (Behr et al., 1994). Schwartz considers the “repeated addition” model of multiplication as a procedural flaw (1988, p. 47). Smith and Thompson (2008) state:

> Conceiving of and reasoning about quantities in situations does not require knowing their numerical value (e.g., how many there are, how long or wide they are, etc.). Quantities are attributes of objects or phenomena that are measurable; it is our capacity to measure them — whether we have carried out those measurements or not — that makes them quantities (p. 101).

In the present study, the linear quantities associated with the sides of the growing rectangles can be categorized as extensive quantities with basic (linear) measurement unit type (e.g., centimeters, inches, units); however, the areal quantities emerge as extensive quantities possessing product — type — units (e.g., square centimeters, square inches, square units) within the growing rectangle itself.

Behr et al. (1994) developed the *generalized notation for mathematics of a quantity* aiming at theoretical analyses and communication within the research community. They applied these systems in the analysis of additive and multiplicative situations. In the notation for the generalized mathematics of quantity,