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Understanding the integral: Students' symbolic forms

Steven R. Jones *

Curriculum and Instruction, University of Maryland, 2311 Benjamin Building, College Park, MD 20742, United States

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ABSTRACT

Researchers are currently investigating how calculus students understand the basic concepts of first-year calculus, including the integral. However, much is still unknown regarding the *cognitive resources* (i.e., stable cognitive units that can be accessed by an individual) that students hold and draw on when thinking about the integral. This paper presents cognitive resources of the integral that a sample of experienced calculus students drew on while working on pure mathematics and applied physics problems. This research provides evidence that students hold a variety of productive cognitive resources that can be employed in problem solving, though some of the resources prove more productive than others, depending on the context. In particular, conceptualizations of the integral as an addition over many pieces seem especially useful in multivariate and physics contexts.

1. Introduction and relevance

In recent decades, more and more attention has been given to compiling a body of research regarding student understanding of mathematics at the undergraduate level. Already this research has provided much information about how students learn and understand a variety of concepts from calculus, differential equations, statistics, and mathematical proof. Among calculus concepts, researchers have focused heavily on student thinking about limits (e.g., Bezuidenhout, 2001; Davis & Vinner, 1986; Oehrtman, Carlson, & Thompson, 2008; Oehrtman, 2004; Tall & Vinner, 1981; Williams, 1991), but have also provided insight about how students understand the derivative (e.g., Marrongelle, 2004; Orton, 1983b; Zandieh, 2000) and Riemann sums and the integral (e.g., Bezuidenhout & Olivier, 2000; Hall, 2010; Orton, 1983a; Rasslan & Tall, 2002; Sealey & Oehrtman, 2005; Sealey & Oehrtman, 2007; Sealey, 2006; Thompson & Silverman, 2008; Thompson, 1994). Overall, the concepts of the derivative and the integral are less explored than the idea of the limit. While the limit is fundamental to calculus, the derivative and the integral have additional layers of meaning above and beyond the limit, as well as meanings that do not necessarily require accessing the concept of a limit (Marrongelle, 2004; Sealey & Oehrtman, 2007; Thompson & Silverman, 2008; Zandieh, 2000). Thus, the derivative and the integral need special attention in order to learn how students understand the main ideas of first-year calculus.

In particular, students' understanding of the integral is an especially valuable topic, since integration serves as the basis for many real world applications and subsequent coursework (Sealey & Oehrtman, 2005; Thompson & Silverman, 2008). The integral shows up in a variety of contexts within physics and engineering (Hibbeler, 2004, 2006; Serway & Jewett,

* Corresponding author. Present address: Sciences and Mathematics Division, Sierra College, V-313B, 5000 Rocklin Road, Rocklin, CA 95765, United States. Tel.: +1 916 660 7987.
E-mail address: sjones3@sierracollege.edu

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2008; Tipler & Mosca, 2008) and students who continue into further calculus courses will encounter the integral more often than the derivative (Salas, Hille, & Etgen, 2006; Stewart, 2007; Thomas, Weir, & Hass, 2009). However, an overreliance on certain interpretations of the integral, such as an "area under a curve," can limit the integral's applicability to these other areas (Sealey, 2006). Evidence of student difficulties with the integral has been documented over the years in several studies (Bezuidenhout & Olivier, 2000; Orton, 1983a; Rasslan & Tall, 2002; Tall, 1992; Thompson, 1994). Additionally, researchers have noted the perception among educators that students transitioning into science courses are routinely struggling to apply their mathematical knowledge to the science domain (Fuller, 2002; Gainsburg, 2006; Redish, 2005). This should be of primary concern for instructors of first-year calculus due to its nature as a service course and the large portion of science students enrolled in these classes (Ellis, Williams, Sadid, Bosworth, & Stout, 2004; Ferrini-Mundy & Graham, 1991).

Hall (2010) demonstrated several ways that students may interpret the definite and indefinite integral, including "area," "Riemann sums," "evaluation," and "language." Conceptions of the integral as an area or as a calculation appeared predominant among his students. Hall's main focus, however, was on the influence of informal language on students' thinking about the integral and he did not attempt to analyze the composition of their concept images (see Tall & Vinner, 1981). Sealey and others, on the other hand, primarily emphasized students' conceptualization of the integral as a Riemann sum (Engelke & Sealey, 2009; Sealey & Oehrtman, 2005, 2007; Sealey, 2006). These studies focus on how students connected the Riemann sum to concepts like the limit and how students used it in solving certain problems, such as approximating the force on a dam. Much of the work was centered on how ideas of accumulation and error were entwined with the conception of the Riemann sum.

This body of work provides insight into a few pieces of students' overall concept images of the integral as well as what is being done to develop students' understanding of accumulation. However, it still leaves open the need to identify the actual cognitive structures that inhabit students' minds regarding the integral. If a student perceives the integral as an area or a Riemann sum, or in some other way, what does that knowledge look like per se? What ideas do the symbols of the integral evoke in students' minds? What aspects of a problem cue students into activating a particular interpretation of the integral? There is still little we know about the meaning students place on the various pieces of the integral symbol structure and how these pieces come together to form the overall concept in a student's cognition.

The purpose of this paper is to document cognitive resources of the integral that students hold and draw on in mathematics and physics contexts. In the next section, the reader is acquainted with the theoretical constructs of *cognitive resources* and a particular type of cognitive resource called a *symbolic form*. The emergent symbolic forms presented in this paper are analyzed for their impact on student thinking during mathematics and physics problems.

2. Theoretical perspective and framework

2.1. Cognitive resources and framing

In this study, the aspect of *knowledge* that is considered is that of *cognitive resources* (Hammer, 2000). Cognitive resources are "fine-grained" elements of knowledge in a person's cognition (Elby & Hammer, 2010). As an example, a student's *concept* of integration may not be a single entity, but may rather be made up of smaller units, including ideas of area, anti-derivatives, summations, or differentials. Each of these may be made up of even smaller units. If this is the case, one cannot claim that a student's *concept* of integration is one fixed object in their cognition. To illustrate, suppose a student properly calculates the integral $\int_0^1 x^2 dx$ using an anti-derivative but then fails to adequately interpret the corresponding Riemann sum $\lim_{n\to\infty}\sum_{k=1}^{n}(x_k)^2\Delta x$. Calling this student's *conception* of integration either correct or incorrect may be too simplistic a view of his or her knowledge (Clement, Brown, & Zietsman, 1989; diSessa, 1993). Within the framework of cognitive resources, inadequate reasoning "differs from the notion of a 'misconception,' according to which a student's incorrect reasoning results from a single cognitive unit, namely the 'conception,' which is either consistent or inconsistent with expert understanding" (Hammer, 2000, p. 53). Instead, it may be the selection of certain cognitive resources over others that results in the satisfactory or unsatisfactory reasoning (Elby & Hammer, 2010; Hammer, Elby, Scherr, & Redish, 2005).

It is important to note that this investigation does not directly study students' *beliefs* about mathematics (Elby & Hammer, 2001), though it is acknowledged that students' beliefs impact the ways in which they might draw on their cognitive resources during problem solving through the process of *framing* (see Lunzer, 1989; MacLachlan & Reid, 1994). Framing means "a set of expectations an individual has about the situation in which she finds herself that affect what she notices and how she thinks to act" (Hammer et al., 2005, p. 97). Framing directly influences students' tacit "selection" of cognitive resources during problem solving and is consequently a key component of interpreting the data in this study.

For the purposes of this paper, a cognitive resource that relies on physical phenomena, such as position or movement, does not count as being purely mathematical and would instead be considered a *blend* of mathematical and physical knowledge (Bing & Redish, 2007; Fauconnier & Turner, 2002). One type of blend that is important to this study is seen in the *symbolic form*, which is described subsequently.