Journal of Mathematical Behavior 34 (2014) 1–13

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Evolving a three-world framework for solving algebraic equations in the light of what a student has met before

David Tall[a,*], Rosana Nogueira de Lima[b], Lulu Healy[b]

[a] *University of Warwick, UK*
[b] *Bandeirante University of São Paulo, Brazil*

ARTICLE INFO

Article history:
Available online 6 February 2014

Keywords:
Theories of learning
Solving equations
Quadratic equations
Procedural embodiment
Three worlds of mathematics

ABSTRACT

In this paper we consider data from a study in which students shift from linear to quadratic equations in ways that do not conform to established theoretical frameworks. In solving linear equations, the students did not exhibit the 'didactic cut' of Filloy and Rojano (1989) or the subtleties arising from conceiving an equation as a balance (Vlassis, 2002). Instead they used 'procedural embodiments', shifting terms around with added 'rules' to obtain the correct answer (Lima & Tall, 2008). Faced with quadratic equations, the students learn to apply the formula with little success. The interpretation of this data requires earlier theories to be seen within a more comprehensive framework that places them in an evolving context. We use the developing framework of three worlds of mathematics (Tall, 2004, 2013), based fundamentally on human perceptions and actions and their consequences, at each stage taking into account the experiences that students have 'met-before' (Lima & Tall, 2008; McGowen & Tall, 2010). These experiences may be supportive in new contexts, encouraging pleasurable generalization, or problematic, causing confusion and even mathematical anxiety. We consider how this framework explains and predicts the observed data, how it evolves from earlier theories, and how it gives insights that have both theoretical and practical consequences.

1. Empirical data and theoretical frameworks for the solution of linear equations

It is our view that theories of learning evolve over time as phenomena are noticed and formulated in coherent ways that later need to take new data into account. In this way initial ideas may be enriched and become part of a more comprehensive whole. In this paper, specific data in linear equations and the transition to quadratic equations will be placed in a broader framework for cognitive development that brings together several distinct strands of research within a single theory.

The research of Filloy and Rojano (1989) suggested that an equation such as $3x - 1 = 5$ with an expression on the left and a number on the right is much easier to solve symbolically than an equation such as $3x + 2 = x + 6$. This is because the first can be 'undone' arithmetically by reversing the operation 'multiply by 3 and subtract 1 to get 5' by adding 1–5 to get $3x = 6$ and then dividing 6 by 3 to get the solution $x = 2$. Meanwhile the equation $3x + 2 = x + 6$ cannot be solved by arithmetic undoing and requires algebraic operations to be performed to simplify the equation to give a solution. This phenomenon is called 'the didactic cut'. It relates to the observation that many students see the 'equals' sign as an *operation*, arising out of experience in arithmetic where an equation of the form $3 + 4 = 7$ is seen as a dynamic operation to perform the calculation, 'three plus four *makes* 7', so that an equation such as $3x - 1 = 5$ is seen as an operation which may possibly be solved by arithmetic 'undoing' rather than requiring algebraic manipulation (Kieran, 1981).

* Corresponding author.
E-mail address: david.tall@warwick.ac.uk (D. Tall).

http://dx.doi.org/10.1016/j.jmathb.2013.12.003

Lima and Healy (2010) classified an equation of the form 'expression = number' as an *evaluation* equation, because it involved the numerical evaluation of an algebraic expression where the input value of x could be found by numerical 'undoing', and more general linear equations as *manipulation* equations, because they required algebraic manipulation for their solution.

On the other hand, if the solution of linear equations is considered in terms of the conceptually embodied notion of a 'balance', the difficulty of the equations is reversed. The equation $3x+2=x+6$ can easily be solved as a balance by imagining the xs to be identical unknown objects of the same weight and representing the equation with 3 xs and 2 units on the left and one x and 6 units on the right. It is then possible to remove 2 units from either side to retain the balance as $3x=x+4$, and then remove an x from both sides to get $2x=4$, leading to $x=2$. In writing the prophetic paper entitled '*the balance model: hindrance or support for the solving of linear equations with one unknown*', Vlassis (2002) noted that, as soon as negative quantities or subtraction are involved, then the embodiment becomes more complicated and hinders understanding. For instance, the equation $3x-1=5$ cannot be represented directly as a balance because the left-hand side $3x-1$ cannot be imagined as $3x$ with 1 taken away if the value of x is not known.

This reveals that the didactic cut and the balance model give rise to very different orders of difficulty. In the didactic cut the equation $3x-1=5$ is easier to solve than the equation $3x+2=x+6$, but in the balance model the order of difficulty is reversed.

The data of Lima and Tall (2008) presented an analysis of Brazilian students' work with linear equations that did not fit either the didactic cut or the balance model. Their teachers had used an 'expert-novice' view of teaching and had introduced the students to the methodology that they, as experts, found appropriate for solving equations, using the general principle of 'doing the same thing to both sides' to simplify the equation and move towards a solution. However, when interviewed after the course, students rarely used the general principle. They did not treat the equation as a balance to 'do the same thing to both sides', nor did they show any evidence of the didactic cut.

Instead, they focused more on the specific actions that they performed to shift symbols around and 'move towards a solution' using two main tactics:

1) 'swop sides, swop signs'
in which an equation $3x-1=3+x$ is operated upon by shifting the 1 to the right and the x to the left and changing signs to get:

$$3x-x=3+1$$
$$2x=4.$$

2) 'swop sides and place underneath'

in which the 2 associated with the expression $2x$ in the equation above is moved from one side of the equation to the other, then placed underneath to give:

$$x=\frac{4}{2}=2.$$

In an attempt to use such rules, some students made mistakes, such as changing $2x=4$ to:

$$\text{(a)}\quad x=4-2 \quad \text{(b)}\quad x=\frac{4}{-2} \quad \text{(c)}\, x=\frac{2}{4}.$$

In (a) the 2 is passed over the other side and its sign is changed; (b) correctly 'shifts the 2 over and puts it underneath' but also 'swops the sign'; (c) shifts the 2 over and puts the 4 underneath. When questioned, *no* student mentioned the principle of 'doing the same thing to both sides', instead they developed what Lima and Tall called *procedural embodiments* which involved embodied actions on the symbols to 'pick them up' and 'move them to the other side' with an extra 'magic' principle such as 'change signs' or 'put it underneath' to 'get the right answer'. Procedural embodiments worked for some students but they also proved to be fragile and misremembered by many others, leading to the wide range of errors that are well known in the literature (Matz, 1980; Payne & Squibb, 1990).

Our purpose is not simply to find and catalogue errors. Instead we seek to evolve a single theoretical framework that covers all three aspects: the didactic cut, the balance model and the problem with 'doing the same thing to both sides'. Such a theoretical framework should relate to both cognitive development and the emotional effects of the learning experience. To integrate these different aspects into a single framework, we begin with a theoretical construct that relates current learning to previous experience.

2. Supportive and problematic met-befores

The effect of previous experience on current learning may be studied using the notion of 'met-before', which has a working definition as 'a structure we have in our brains *now* as a result of experiences we have met before' (Lima & Tall,