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Students' and experts' schemes for rate of change and its representations

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ABSTRACT

The purpose of this paper is to articulate students' and mathematicians' schemes for rate of change and describe how those schemes affected their conception of rate of change in graphical, tabular, and algebraic representations. We argue that experts' schemes for rate of change allowed them to conceive of rate of change as similar across representations while students' conceptions of rate of change depended on a single representation resulting in their inability to conceive of rate of change represented in multiple ways. We describe the difficulties that students had in conceiving the rate of change represented in multiple ways and contrast those with the coherence that mathematicians possessed. Lastly, we discuss the implications of a coherent scheme of meanings for the use of multiple representations. We propose that exposure to multiple representations requires identification of the meanings, which we intend the students to develop before considering various ways in which to represent that meaning.

1. Introduction

1.1. Multiple representations: central themes

Why our interest in multiple representations? First, researchers have found that articulating a concept using multiple representations during instruction seems to help students solidify their understanding of that concept (Ainsworth, 2006; Arcavi, 2003; Goerdt, 2007; Martinez-Planell & Trigueros, 2009; Yerushalmy, 1991, 1997, 2008). In this case, the coherence between representations lies with the instructor, who intends to communicate that same understanding to the student. This issue extends to much of multiple representations literature, which often takes the perspective of an expert who already possesses a coherent scheme of meanings with which to interpret multiple representations (Ainsworth, 1999; Brenner et al., 1997; DeMarois & Tall, 1996; Janvier, 1987; Rider, 2004; Yerushalmy, 1997). However, in many cases, what students represent is different from what experts might expect. Examples range from algebra (Yerushalmy, 1991) to covariation (Moore, Paoletti, & Musgrave, 2013) to many other areas of K-12 and undergraduate mathematics. Because much of the existing research comes from the perspective of experts who already can identify coherence between representations, there has been little attention given to describing the incompatibility that students perceive between representations and moreover, what the students *believe* they are representing.

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1.2. Multiple representations over time: focus on function

The focus on multiple representations is not new, but a focus on what students believe they are representing was not a part of the attention to representations in the 1980s. During that time, researchers began to study the importance of representations and symbolization. The link between symbolizations in representations became the basis for a discussion about multiple representations of a mathematical idea as a basis for instruction and learning (Clement, 1985; Davis, 1983; Janvier, 1985, 1987; Janvier, Girardon, & Morand, 1993; Kaput, 1985). While these researchers provide compelling accounts of experts drawing coherence between representations and students' inability to do the same (a deficit model), the nature of the research questions constrained them from considering what it was the students thought they were representing. This criticism was particularly important for the case of function, as illustrated by Thompson and Sfard (Thompson, 1994b, 2013; Thompson & Sfard, 1994):

> Tables, graphs, and expressions might be multiple representations of functions to us, but I have seen no evidence that they are multiple representations of anything to students. In fact, I am now unconvinced that they are multiple representations even to us, but instead may be . . . areas of representational activity among which we have built rich and varied connections. . . It may be wrongheaded to focus on graphs, expressions, or tables as representations of function, but instead focus on them as representations of something that, from the students' perspective, is representable, such as some aspect of a specific situation. The key issue then becomes twofold: (1) to find situations that are sufficiently propitious for engendering multitudes of representational activity and (2) orient students to draw connections among their representational activities in regard to the situation that engendered them (Thompson, 1994b, pp. 39–40)

There are at least two key ideas from Thompson's argument. First, if students are to conceive of something represented in multiple ways, they must conceive of what is to be represented (i.e. a function, a rate of change) and understand the constraints of the representational system so they can represent what they intend. Second, it is important to consider if it is possible for experts (in this case, mathematicians) to conceive of something represented in multiple ways, or if, as Thompson says, experts build 'rich and varied connections' between representations using representational activity. Indeed, subsequent frameworks for representation allowed for a representation to include a "correspondence"—wherein two conceptual entities are related to each other in such a way that one is taken to represent the other (Goldin, 1998; Goldin & Kaput, 1996; Kaput, 1998). The idea of correspondence, fundamentally focused on what the individual perceives, allowed for consideration of what it was that students represent as a basis for multiple representations (Kaput, Blanton, & Moreno-Armella, 2008).

We think the issues raised in the discussion about multiple representations and the attention to what it is students represent brings up a key distinction that has not been fully addressed in literature. Specifically, we question if students develop a single scheme for a concept and conceive of that scheme applied in multiple mediums, or if students have uncoordinated schemes for a concept that depend on the representation in which the student is operating. For example, if a student conceives of a function as a rule when defined algebraically and as a picture in a graphical context, we would hypothesize that he/she has multiple schemes for the function concept, each of which is tied to a particular external representation. In this case, the student would not be representing the same concept across representations even though he/she names the object 'function' in each case. In another case, the student might conceive of a function as a picture and apply that scheme to every type of representation, whether sensible or not. We think that the issues we have raised for functions and multiple representations also apply to rate of change, which is the focus of this paper.

It is somewhat surprising that more attention has not been given to rate of change when so much has been afforded to multiple representations of function when rate of change itself can be thought about as a function. Rate of change and representations of it are crucial to students as they progress through calculus and other subject areas (physics, engineering, natural sciences) yet we do not know enough about how students think about rate of change to argue that the results from the function literature naturally apply to this setting. Moreover, the issues we raised above put into question whether there are differences between how experts and novice students conceive of multiple representations, particularly for rate of change. This paper explores these themes.

1.3. Purpose and research question

The purpose of this paper is to characterize students' and experts' schemes for rate of change and how they represent those meanings using graphical, symbolic, tabular contexts. To do so, we focus on results from a study conducted with college level calculus students and mathematicians about their understanding of rate of change and how they represent rate of change across multiple contexts. The major research question guiding this study was:

What schemes do college level calculus students and mathematicians use to conceive of and represent rate of change in multiple contexts?

In exploring this question, we argue that students' schemes to solve rate of change problems were abstracted from a particular representation, and were not powerful/general rate of change schemes. This meant that they could not use their scheme to create meaning across multiple ways of representing rate of change. This brought into focus another issue: though