Journal of Mathematical Behavior 34 (2014) 99–113

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Identifying discursive entry points in paired-novice discourse as a first step in penetrating the paradox of learning mathematical proof

Katherine S. Remillard *

Science Outreach Center/Mathematics Department, Saint Francis University, P.O. Box #600, Loretto, PA 15940-0600, USA

ARTICLE INFO

Article history:
Available online 15 March 2014

Keywords:
Commognition
Discourse
Discursive entry points
Learning-as-participation
Learning paradox
Proof

ABSTRACT

In this article, I demonstrate the efficacy of using the commognitive research framework to study the learning of mathematical proof. To this end, I present a detailed analysis of an excerpt of paired discourse by undergraduate mathematics majors working on an introductory proof task. Additionally, I draw on the excerpt to illustrate that there exists in discourse between novice interlocutors natural opportunities, especially ripe with potential, in which experts could intervene to steer the discourse toward increasing mathematical sophistication. I have named these opportunities *discursive entry points* and suggest their identification as significant in addressing the learning paradox as it relates to mathematical proof. Overall, the findings illustrate how rich description of the nature of learning in paired discourse, a result of the commognitive research lens, can inform reform era instructional practices.

1. Introduction

The seminal work of Harel and Sowder (1998, 2007) demonstrated a problematic prevalence of the authoritative proof scheme among students of mathematics. Characteristic of this scheme is an over-reliance on the teacher and text. Students expect to be told solution procedures. They ask for help on problems without a serious effort to solve them on their own first. And they are reluctant to inquire about the motivation and reasoning behind given mathematical assertions. The authoritative scheme, the predominant practice of students, is inconsistent with the creative practices of mathematicians. Sriraman (2004), for example, found that preceding proof construction, mathematicians frequently engage in social interaction, imagery, heuristics, and intuition. Weber (2008) also found mathematicians using a variety of strategies to validate proof, including formal reasoning, the construction of rigorous proofs, informal deductive reasoning, and example based reasoning. Moreover, the mathematicians' conceptual knowledge, the mathematical domain of the proof, and the status of the proof's author were important factors in validation. Such a contrast between learning a domain and the practice of the domain's experts implores us to consider how instruction fosters (or does not foster) the authoritative scheme. The "conflict between the practice of mathematicians on the one hand, and their teaching methods on the other, produces problems amongst students" (Alibert & Thomas, 1991, p. 215). In particular, student exposure to mathematics has often been limited to its finished product form. Missing from their experience is an active, albeit messy role in the construction of knowledge. As such, understanding how

* Tel.: +1 814 472 6772.
E-mail addresses: kremillard@francis.edu, katewallace78@hotmail.com

http://dx.doi.org/10.1016/j.jmathb.2014.02.002
0732-3123/

students construct mathematical knowledge, that is move from informal, often intuitive, understandings to more formal ones, is of increasing interest to researchers in undergraduate mathematics education in general (Rasmussen, Zandieh, King, & Teppo, 2005) and for proof learning in particular (Raman, 2003; Selden and Selden, 2003; Weber, 2001; Zandieh, Larsen, & Nunley, 2008).

Here we discuss two theoretical constructs that aid the discussion on bridging the gap between novice learners' informal understandings of mathematics and more rigorous ones. First, Rasmussen et al. (2005), demonstrate the utility of the notion of *advancing mathematical activity* in describing an evolution in undergraduate students' reasoning for the mathematical practices of symbolizing, algorithmatizing, and defining. The progression of reasoning can be understood as interplay between *horizontal mathematizing* and *verticial mathematizing*. Horizontal mathematizing (experimenting, pattern snooping, classifying, conjecturing, organizing) lays the groundwork for vertical mathematizing (reasoning about abstract structures, generalizing, and formalizing) which in turn serves as context for further horizontal mathematizing. For proof, Raman (2003) distinguishes between the *private argument*, one that engenders understanding, and the *public argument*, that which has sufficient rigor to convince a particular mathematical community. Interestingly, for mathematicians (experts) the two arguments are inextricably linked. Students, however, fail to see an essential connection between their privately held ideas and the formal, public proof they are aiming to produce. To better understand this difference between mathematicians and students, Raman identifies three kinds of ideas used in proof production and evaluation: heuristic, procedural and key. A *heuristic idea* is based on informal understanding and gives the sense that something ought to be true. A heuristic idea operates as a private argument. It does not, however, lead to a formal proof. A *procedural idea*, based on logic and formal manipulations, leads to a formal proof or public argument though it does not provide understanding. Of interest here is the *key idea*, one which is heuristic but can be mapped to a formal proof. Key ideas, which address the "why" of the claim, provide a bridge between private and public arguments. Mathematicians value and make use of key ideas, but students often lack them.

The challenge, then, is to assist students in the active construction of their own knowledge. Recent curricular reforms emphasize student mastery of mathematical communication for this purpose (Barker et al., 2004; CCSSI, 2010; NCTM, 2000). Growing support exists within practitioner and theoretical research for a pedagogical shift from lecture-based classrooms (univocal) to those utilizing the student voice (dialogic) (King, 2001; Knuth & Peressini, 2001). This raises important questions about the teacher's role as it relates to the nature of learning. Krussel, Edwards, and Springer (2004) call the deliberate actions taken by a teacher to mediate, participate in, or influence the discourse in mathematics classrooms *teacher discourse moves.* The teacher has the responsibility to manage the content of discourse, choosing topics and assigning tasks; the structure of discourse, using whole- or small-groups; and the physical and temporal boundaries of discourse, deciding on tools and time available. The teacher's actions within these parameters have important intended and unintended consequences. Accordingly, researchers have focused intently on discourse moves.

The work of Harel and Rabin (2010), for example, examined teaching practices that contribute to students' understanding of the instructor as the sole arbiter of mathematical correctness. They identified nine practices believed to be linked to the authoritative proof scheme. Two examples are: (a) answering students' questions mainly by telling how to perform a task or whether the answer is correct and (b) providing social justifications rather than intellectual justifications. Other research has highlighted the complex demands placed on teachers trying to adopt dialogic classroom discourse. The challenge of how to encourage increased student participation in classroom discussion while maintaining mathematical precision and productivity has been well documented (Nathan & Knuth, 2003; Sherin, 2002). Sherin identified the "filtering approach" as a way for the teacher to balance these tensions. First, the instructor elicits a wide range of student responses. Then, he or she focuses on the important mathematical ideas within these responses. The research of Blanton, Stylianou and David (2003) indicates that students can make gains in their ability to construct proofs when they participate in well-scaffolded whole-class discussions. The researchers identified two effective scaffolds: (a) *facilitative utterances* wherein the teacher revoices and confirms student-originated ideas and (b) *transactive prompts* in which the teacher repeatedly requests clarification, justification, elaboration, and critique. Finally, Larsen and Zandieh (2007) have demonstrated the potential efficacy of using Lakatos' constructs of monster-barring, exception-barring, and proof-analysis for making sense of students' classroom mathematical activity. They suggest that the constructs may be additionally valuable as heuristics for instructional design, especially in the area of realistic mathematics education which has as its primary characteristic the guided reinvention of mathematics.

In this era of educational reform, teachers are increasingly encouraged to use non-lecture formats in their classrooms. The research literature on *teacher* discourse moves in whole-class discussions related to teaching mathematics in general and proof in particular is growing more robust. What about the mathematical productivity of small-group discourse? This article is concerned with discourse *between students* and what affordances, if any, it might provide for advancing mathematical activity.

2. Framework

2.1. Commognition

I draw on Sfard's (2008) theory of commognition for a theoretical and methodological framework. The neologism combines two words: communication and cognition. Sfard stresses that the processes of thinking (individual cognition) and