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Understanding multidigit whole numbers: The role of knowledge components, connections, and context in understanding regrouping 3+-digit numbers

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ABSTRACT

This case study of a PST's understanding of regrouping with multidigit whole numbers in base-10 and non-base-10 contexts shows that although she seems to have all the knowledge elements necessary to give a conceptually based explanation of regrouping in the context of 3-digit numbers, she is unable to do so. This inability may be due to a lack of connections among various knowledge components (conceptual knowledge) or a lack of connections between knowledge components and context (strategic knowledge). Although she exhibited both conceptual and strategic knowledge of numbers while regrouping 2-digit numbers, her struggles in explaining regrouping 3-digit numbers in the context of the standard algorithms indicate that explaining regrouping with 3-digit is not a mere extension of doing so for 2-digit numbers. She also accepts an overgeneralization of the standard algorithms for subtraction to a time (mixed-base) context, indicating a lack of recognition of the connections between the base-10 contexts and the standard algorithms. Implications for instruction are discussed.

Consider subtracting 23 − 7 using the standard regrouping algorithm used in the United States. To do so, one would regroup 1 ten into 10 ones (see Fig. 1a and b), this regrouping is correct because the value of the regrouped digit stays the same; however, the reference unit of the digit changes: one *ten* is reconceived as 10 *ones*. Thus, to explain the mathematics underlying this regrouping, one would need to know the following: (a) that the 2 represents 2 tens (or 20) and the 3 represents 3 ones; (b) that 1 ten can be reconceived as 10 ones (or know that 20 can be separated into 10 and 10); and (c) that 10 ones can be moved from the ten's place to the one's place and combined with the 3 ones already there.

Preservice teachers (PSTs) struggle explaining the value of regrouped digits in the context of addition and subtraction (Ball, 1988; Ma, 1999; Southwell & Penglase, 2005; Thanheiser, 2009), and although some PSTs may interpret digits correctly at some times, they may interpret digits incorrectly at others (Thanheiser, 2009, 2010). Research results indicating that adults have trouble understanding and explaining digits in a number are often met with surprise and sometimes with disbelief. The reader may share this disbelief because after one understands the topic of whole numbers and regrouping, he or she may be unable to imagine not having this understanding. In previous work (Thanheiser, 2009, 2010), I have demonstrated the complexities inherent in explaining the values of the regrouped digits in the context of addition and subtraction (a

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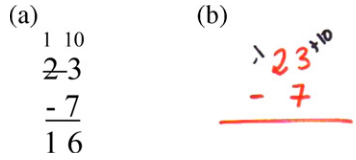

Fig. 1. (a) Subtracting 23 − 7 using the standard algorithm. (b) One PST's regrouping in the context of 23 − 7.

review follows below). In this paper, I focus on examining the connections between various types of knowledge, namely *procedural*, *conceptual* (cp. Hiebert & Lefevre, 1986), and *strategic* (cp. Weber, 2001) knowledge. More specifically, I examine whether knowing the individual components of knowledge needed to explain a concept such as regrouping (as described above) enables one to explain that concept and how knowledge of a concept relates to understanding in what situations it is applicable.

1. Background

Research to date has shown that children as well as teachers and PSTs struggle understanding numbers. Children struggle when learning about number and the numeration system. Only 60% of U.S. 8th graders were able to write a 3-digit number when given digits and conditions related to their place values (Kouba & Wearne, 2000). And only about half the children in fourth grade understand that the 1 in 16 represents 10 (Kamii, 1986; Ross, 1990). Children often build juxtaposed and unrelated systems of ones and tens (Cobb & Wheatley, 1988; Kamii, 1986; Ross, 1990) and, thus, struggle to relate tens and ones. For these children, tens and ones are just different unit types: "Ten was, for them, one thing which was not itself composed of units" (Cobb & Wheatley, 1988, p. 1). These children, therefore, struggle to understand numbers because they do not understand the basic relationships among the digits in the number. To understand 2-digit numbers, students must relate tens and ones (Fuson et al., 1997; Kamii, 1986).

When children learn the standard algorithms, they are often drilled on the procedures, and many fail to construct meaning for the algorithms. Learning procedures without understanding can be detrimental to children (Ball, 1988; Kamii, 1986). Children learn where to write the digits but might not understand why when they add 7 and 5, for example, they write the 2 below and regroup the 1 to the next column. Without constructing meaning for the algorithms, children might concentrate more on the rules of the algorithm than on "the essence of the numeration system—the numerals have different *values* depending on their *place*" (Ball, 1988, p. 49).

To summarize, children struggle understanding multidigit whole numbers, and although they may be able to apply procedures, they are unlikely to be able to explain the underlying mathematics or why the procedures are applicable in a context. For children to form a conceptual and strategic understanding of mathematics rather than to develop only procedural skills, they need to be taught in a way that helps them construct meaning and connect that meaning to contexts. To construct meaning for most mathematical concepts, children need to have a conceptual understanding of number and need to be able to draw on that knowledge in appropriate contexts (strategic knowledge). To guide students toward such an understanding of number, teachers, in turn, need a profound understanding (Ma, 1999) of number.

Teachers and PSTs also struggle with understanding numbers. PSTs can perform algorithms but struggle when asked to explain them (Ball, 1988; Ma, 1999; Southwell & Penglase, 2005; Thanheiser, 2009, 2010). Although the PSTs are able to execute the procedures, they often do so by merely manipulating algorithms (McClain, 2003; Tirosh & Graeber, 1989).

Several researchers have focused on various aspects of PSTs' understanding of numbers: number sense (Graeber, Tirosh, & Glover, 1989; Huang, Liu, & Lin, 2009), alternative algorithms (Luo, 2009), and number concepts (Southwell & Penglase, 2005). These researchers have shown that PSTs tend to focus on procedures and that place value seems to cause the most difficulty for PSTs.

Because PSTs are usually satisfied with their own procedural understandings (until they are asked to explain the underlying mathematics), they are likely to take their own procedural fluency as evidence of conceptual understanding (Graeber, 1999). One PST reflected,

> I remember feeling very frustrated [when asked why regrouping works in the context of addition and subtraction] because I couldn't explain how I solved the addition and subtraction problems. To me, those kinds of problems you just solved. I don't remember ever having to explain why I did a certain step or math procedure. (PST in elementary mathematics methods course)

Procedural fluency will not, however, prepare PSTs to teach in a way that fosters conceptual understanding or strategic knowledge in the children they teach.

The issue of the kinds of knowledge teachers need to teach with a focus on developing sense making and conceptual understanding has been the focus of much research in the mathematics education community (e.g., Hill, Ball, & Schilling, 2008; Hill, Rowan, & Ball, 2005; Ma, 1999; National Research Council, 2001; Shulman, 1986; Silverman & Thompson, 2008) and has resulted in various frameworks describing *mathematical knowledge for teaching*. All descriptions/frameworks include mathematical content knowledge as part (or center) of the frameworks. A consensus seems to exist that mathematical