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An operational definition of learning

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ABSTRACT

An operational definition offered in this paper posits learning as a multi-dimensional and multi-phase phenomenon occurring when individuals attempt to solve what they view as a problem. To model someone's learning accordingly to the definition, it suffices to characterize a particular sequence of that person's disequilibrium–equilibrium phases in terms of products of a particular mental act, the characteristics of the mental act inferred from the products, and intellectual and psychological needs that instigate or result from these phases. The definition is illustrated by analysis of change occurring in three thinking-aloud interviews with one middle-school teacher. The interviews were about the same task: "Make up a word problem whose solution may be found by computing 4/5 divided by 2/3."

> An *operational definition* is a showing of something—such as a variable, term, or object—in terms of the specific process or set of validation tests used to determine its presence and quantity. Properties described in this manner must be publicly accessible so that persons other than the definer can independently measure or test for them at will. An operational definition is generally designed to model a *conceptual definition* (Wikipedia)

1. Introduction

Conceptual definitions of *learning* successfully convey the messages about different theoretical perspectives on learning, but suffer from the lack of operability. For instance, many frequently used in mathematics education discourse conceptualizations of learning, such as *learning as acquisition*, *learning as participation*, *learning as problem solving* or *learning as assimilation and accommodation*, refer to the key processes involved, but are insufficient in order to operationally capture the essence of the intended change (Sfard, 1998; Skinner, 1950; Von Glasersfeld, 1995).

The need to build research programs on *operational* definitions of learning is recognized by many scholars (e.g., diSessa & Cobb, 2004; Siegler, 1996; Simon et al., 2010; Steffe, 2003). Simon et al. (2010) distinguish between two main operational approaches to studying learning. The first one focuses on fostering perturbations and characterizing the students' learning trajectories by specifying a resulting series of understandings or conceptual steps through which students pass. This approach is consonant with Von Glasersfeld's (1995) description of Piaget's theory of learning, and is exemplified in Simon's et al. (2010) paper by detailed discussion of several studies utilizing micro-generic analysis of students' reasoning while solving challenging for them arithmetical tasks (Steffe & Thompson, 2000; Steffe, 2003). The second approach – presented as recently emerging from Martin Simon's and his colleagues own research – focuses on examination of the *process* by which students

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progress from one of these understandings or conceptual steps to a subsequent one. Fostering perturbation is a possible but not necessary part of this approach. Simon et al. (2010) describe the main differences between the approaches as follows:

> In an approach in which the emphasis is on promoting perturbation, the data generally reveal the *result*, the successful solution to the new problem. Because previously the student was unable to produce a solution of this type and now is able to, there may be little or no data on the learning (change) process. In our approach, the researchers use a sequence of tasks to engender particular activity on the part of the students that foster the intended learning. If the task sequence is successful, the researchers are able to observe the students' activity over the course of the entire task sequence. This provides data on the *process* by which the new learning comes about, that is, evidence of modifications in the students' *thinking* [italics added] as a function of their mathematical activity as they participate in the sequence of tasks (Simon et al., 2010, pp. 107-108).

The scholars note that the approaches are rather complementary than antagonistic, and may highlight different aspects of learning as modeled through different types of situations. Our goal is to take this idea a step further and offer an operational definition of learning which would balance the two approaches. Note: we do not suggest in this paper a *conceptually* new definition of learning, but offer instead an *operational* approach to modeling change in individual's knowledge structures over time in a way accounting for both processes and products of the change. The suggested approach is rooted in Piaget's theory of equilibration and is asserted in terms of a particular conceptual framework, called *DNR-based instruction in mathematics* (Harel, 2008a, 2008b, 2008c). The selected elements from DNR are outlined in Section 2. In Section 3, we illustrate the approach by discussing three episodes with one middle-school teacher. The episodes are about the teacher's attempts to solve the same task: "Make up a word problem whose solution may be found by computing 4/5 divided by 2/3." We conclude, in Section 4, with a discussion of the operational nature of the presented definition of learning and its possible contribution to the existing literature on the topic.

2. DNR-oriented definition of learning

"Learning" in DNR is operationally defined as a continuum of disequilibrium–equilibrium phases manifested by (a) *intellectual needs* and *psychological needs* that instigate or result from these phases and (b) *ways of understanding* or *ways of thinking* that are utilized and newly constructed during these phases. The italicized two pairs of terms in this definition are concepts from DNR; they will be discussed below—the first pair in Section 2.3, and the second pair in Section 2.2. These concepts are oriented within a particular set of premises; they will be briefly discussed in Section 2.1. Prior to all this, however, we will describe in few words what DNR is.

DNR is a conceptual framework that stipulates conditions for achieving critical goals such as provoking students' *intellectual need* to learn mathematics, helping them to construct mathematical *ways of understanding* and *ways of thinking*, and assuring that they internalize and retain the mathematics they learn. The framework can be thought of as a system consisting of three categories of constructs: *premises*—explicit assumptions underlying the *DNR* concepts and claims; *concepts* oriented within these premises; and *instructional principles*—claims about the potential effect of teaching actions on student learning justifiable in terms of these premises and empirical observations. The system states three foundational principles: the *duality principle*, the *necessity principle*, and the *repeated-reasoning principle*; hence, the acronym *DNR*. The principles are presented in Section 2.4. Admittedly, here we only discuss those *DNR* elements that are needed for our definition of learning. For an extended discussion of *DNR* see Harel (2008a, 2008b, 2008c).

2.1. Three DNR premises

DNR is based on a set of eight premises, seven of which are taken from or based on known theories. For the concerns of this paper, we only need three premises that concern knowledge and knowing:

Knowledge (of Mathematics): Knowledge of mathematics consists of all *ways of understanding* and *ways of thinking* that have been institutionalized throughout history (Harel, 2008a).
Knowing: Knowing is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium (Piaget, 1985).
Knowing-knowledge linkage: Any piece of knowledge humans know is an outcome of their resolution of a problematic situation (Piaget, 1985).

The Knowledge (of Mathematics) premise concerns the nature of the mathematics knowledge by stipulating that *ways of understanding* and *ways of thinking*—the terms to be defined below—are the constituent elements of this discipline, and therefore instructional objectives must be formulated in terms of both these elements, not only in terms of the former, as currently is largely the case (Harel, 2008a). The next two premises are Piagetian: The Knowing premise is about the mechanism of knowing: that the means—the only means—of knowing is a process of assimilation and accommodation. A failure to assimilate results in a disequilibrium, which, in turn, leads the mental system to seek equilibrium, that is, to reach a balance between the structure of the mind and the environment. The Knowledge-Knowing Linkage premise, too, is inferable