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Studying teachers' mathematical argumentation in the context of refuting students' invalid claims

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ABSTRACT

This study investigates teachers' argumentation aiming to convince students about the invalidity of their mathematical claims in the context of calculus. 18 secondary school mathematics teachers were given three hypothetical scenarios of a student's proof that included an invalid algebraic claim. The teachers were asked to identify possible mistakes and explain how they would refute the student's invalid claims. Two of them were also interviewed. The data were analysed in terms of the content and structure of argumentation and the types of counterexamples the teachers generated. The findings show that teachers used two main approaches to refute students' invalid claims, the use of theory and the use of counterexamples. The role of these approaches in the argumentation process was analysed by Toulmin's model and three types of reasoning emerged that indicate the structure of argumentation in the case of refutation. Concerning the counterexamples, the study shows that few teachers use them in their argumentation and in general they underestimate their value as a proof method.

1. Introduction

A central activity in which mathematics teachers are involved in their everyday practice is to interpret students' responses or queries and to give them mathematically and pedagogically appropriate answers. In particular, when a student makes an invalid claim, the teacher needs to argue by offering a counterexample in order to persuade her of the invalidity of her claim and help her to develop an understanding of the mathematical situation. To give an appropriate example in terms of its mathematical correctness and its pedagogical effectiveness is a very demanding task and is related to the teacher's knowledge, e.g. Ball, Thames, & Phelps, 2008) and to the teacher's example space (Watson & Mason, 2005).

There is an increasing number of studies that investigate the counterexamples that students or teachers give in their attempt to refute an invalid claim. There are studies which examine the difficulties that students encounter concerning the status and the generation of counterexamples or investigate their proof schemes in the case of refutation of a mathematical claim (e.g. Balacheff, 1991; Ko & Knuth, 2009; Larsen & Zandieh, 2008; Lin, 2005; Zaslavsky & Ron, 1998). Similar issues have also been identified in the case of teachers with an emphasis on the mathematical and pedagogical appropriateness of the examples and counterexamples (Barkai, Tsamir, Tirosh, & Dreyfus, 2002; Peled & Zaslavsky, 1997; Potari, Zachariades, & Zaslavsky, 2009; Rowland, 2008; Zaslavsky & Peled, 1996; Zazkis & Chernoff, 2008; Zodik & Zaslavsky, 2008). These studies have produced different typologies to characterize the examples given by the teachers and have also developed our

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knowledge about the process of teachers' choice of examples. Moreover, a number of studies on proof focus on the process of argumentation and attempt to reveal aspects of students' and teachers' reasoning (e.g. Inglis, Mejia-Ramos, & Simpson, 2007; Pedemonte, 2007). However, little emphasis has been given on the process of argumentation while refuting an invalid claim and in particular on the role of the counterexamples in this process. Our study attempts to investigate how teachers refute invalid algebraic claims included in students' written proofs in the context of Mathematical Analysis. In particular, we focused on:

- the content of teachers' argumentation
- teachers' structure of argumentation and the underlying reasoning
- the types of the generated counterexamples.

2. Theoretical background

By studying teachers' argumentation in convincing students about the invalidity of their claims we investigate their mathematical reasoning and beliefs. In developing convincing arguments the teacher has to choose appropriate counterexamples, a process which reveals aspects of her mathematics knowledge for teaching (Ball et al., 2005). Below, we discuss research findings concerning counterexamples and the structure of argumentation that are related to our study.

2.1. Counterexamples as a means of refuting mathematical claims

Examples play a central role in both the development of mathematics as a discipline and in the teaching of mathematics. Bills et al. (2006) distinguish three special descriptive labels that are used for examples: 'generic example', 'counterexample' and 'non-example'. The counterexamples, on which our study focuses, are basic tools for refuting invalid claims and, as Bills et al. (2006) state, they "need a hypothesis or assertion to counter, but they may do this in the context of a concept, a procedure or even (part of) an attempted proof" (p. 127). In mathematics education research various characterizations of the counterexamples have been given. Peled and Zaslavsky (1997) categorized the counterexamples according to their explanatory power into specific, semi-general and general ones. The specific counterexamples only disprove a statement. The semi-general and general counterexamples are transparent as they offer some explanation why the statement is not true as well as ideas about the process of generating counterexamples. Zazkis and Chernoff (2008), by focusing especially on the pedagogical role of counterexamples and in particular on their convincing power, introduced the notions of pivotal and bridging example. These examples aim to confront learners' incorrect mathematical claims creating cognitive conflict (pivotal counterexample) and resolving it (bridging counterexample).

An important issue in the research concerning counterexamples is the meaning that mathematicians, students and teachers have attributed to the counterexamples as a tool for refuting a conjecture or an invalid claim. Lakatos (1976) drawing on the discussion among mathematicians in the 19th century about the Euler theorem concerning the relation between the faces F, the edges E and the vertices V of a polyhedron, described three ways in which mathematicians faced counterexamples. The first way, which was called monster-barring, was to devise a definition of a polyhedron in order to bar the known polyhedra for which the relation $F+E=V+2$ was not true. The second way, which was called exception-barring, was to reduce the class of the polyhedra in the theorem barring the known counterexamples. The third way, which was called proofs and refutation, was to analyze the proof and find the class of the polyhedra for which the proof is valid.

Balacheff (1991) and Larsen and Zandieh (2008) have identified that most types of treatment of a refutation described by Lakatos were also observed in students' work. In particular, Balacheff observed these types in the mathematical activity of 13–14-year old students in the context of geometry, while Larsen and Zandieh in the work of undergraduate students in the context of an algebra course. Lin (2005) also studied 7th through 9th grade students' refutation processes by classifying their arguments. Zaslavsky and Ron (1998) investigated 9th and 10th grade top level students' understanding of counterexamples. Their findings indicated students' difficulties in realizing the status and the role of counterexample as a proof method as well as in generating correct counterexamples.

Concerning prospective mathematics teachers, Ko and Knuth's study (2009) revealed similar problems to those reported by Zaslavsky and Ron (1998). Peled and Zaslavsky (1997) studied pre-service and in-service secondary school mathematics teachers' ability to produce counterexamples. Their study indicated that in-service teachers constructed better counterexamples than pre-service teachers in the context of algebra. Zodik and Zaslavsky (2008) studied the pre-planned and spontaneous examples that experienced secondary school mathematics teachers chose in and for the mathematics classroom in the content of geometry and algebra. In relation to counterexamples, they identified two issues in their analysis; one referred to whether the given counterexample logically contradicted the claim and the other referred to whether it existed. They also investigated the kinds of knowledge teachers need to draw on in producing examples and counterexamples. Potari et al. (2009), by using hypothetical classroom scenarios, focused on secondary school mathematics teachers' reasoning for refuting students' invalid claims in the context of geometry. In this study, the issue of existence of counterexamples also emerged. Moreover, a rather narrow meaning of the theorems and their use to refute invalid claims was manifested.