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A power meaning of multiplication: Three eighth graders' solutions of Cartesian product problems☆

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ABSTRACT

This article examines data from five teaching episodes with three eighth grade students who were participants in a 3-year constructivist teaching experiment. The five teaching episodes were a transition point in the teaching experiment: the students were beginning to work in contexts that were deemed to support their development of a meaning for squaring quantities—a power meaning of multiplication. Prior to these teaching episodes, the students had worked in contexts that were deemed to support their development of a linear meaning of multiplication.

This paper focuses on the novel cognitive operations and multiplicative concepts that the students developed to solve Cartesian product problems, problems that were deemed could support students to establish a power meaning of multiplication. The findings from the study contribute to prior research by (1) examining an appropriate use for Cartesian product problems with middle grades students, and (2) identifying similarities and differences in the multiplicative concepts students constructed to solve linear-meaning multiplication problems and power-meaning multiplication problems. Implications for teaching are considered.

1. Introduction

K-8 curricula primarily contain problems that are aimed at supporting students' development of a linear meaning of multiplication, and significantly less attention is given to problems that could potentially support students to develop non-linear meanings of multiplication (e.g., power and exponential meanings) (Confrey, 1994; Greer, 2010; Van Dooren, De Bock, Janssens, & Verschaffel, 2008). This organizational structure is aligned with the conjecture that the multiplicative concepts that students construct to solve linear-meaning multiplication problems have the potential to support their construction of multiplicative concepts to solve non-linear meaning multiplication problems (cf. Steffe, 1994). However, researchers have argued that one unintended consequence of this curricular structure is that students tend to over-rely on linear meanings of multiplication in situations where such a meaning is not warranted; students anticipate that problem situations they encounter in school will involve a linear meaning of multiplication precisely because the majority of problems in school curricula do (Greer, 1992, 2010; Van Dooren et al., 2008).

Students' development of non-linear meanings of multiplication and their differentiation of them from linear meanings of multiplication becomes increasingly important as students take more advanced mathematics courses in high school.

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However, researchers have found that differentiating between linear and non-linear meanings of multiplication remains difficult for students. For example, researchers investigating high school students' algebraic reasoning have identified that students commonly confuse a linear meaning of multiplication (i.e., doubling, tripling, etc.) and a meaning for multiplication that involves raising a quantity to a whole number power (i.e., squaring, cubing, etc.)—a power meaning of multiplication. This confusion has been expressed in researchers' findings in a range of ways, including that students commonly conclude that "$(x+y)^2=2x+2y$" or that "$(x+y)^2=x^2+y^2$" when working with algebraic symbols (Matz, 1982; Sleeman, 1982), that students overuse squaring in notation for doubling and tripling (MacGregor & Stacey, 1997), and that students frequently represent non-linear polynomial functions as a line through the origin (Leinhardt, Zaslavsky, & Stein, 1990).

These student difficulties, along with the structure of current curricula, indicate that it is important to address the following research questions (RQs):

(RQ1) How are students' multiplicative concepts similar and different when solving linear-meaning multiplication problems and power-meaning multiplication problems?
(RQ2) Can the multiplicative concepts that students develop to solve linear-meaning multiplication problems support their development of multiplicative concepts to solve power-meaning multiplication problems? If so, how?

One way to respond to these questions is to provide models of the similarities and differences in the multiplicative concepts students construct to solve linear- and power-meaning multiplication problems. Such models can then be used to inform discussions about how teachers and curricula can effectively support students to develop and differentiate these two meanings of multiplication.

The purpose of this paper is to address these questions. To accomplish this purpose, I analyze data from a 3-year teaching experiment that was conducted during three students' sixth, seventh, and eighth grade years. During the students' sixth and seventh grade years, they solved problems that I deemed involved a linear-meaning of multiplication (e.g., operations with fractions, linear equations, ratios, and constant rates) (see Hackenberg, 2007, 2010; Hackenberg & Tillema, 2009). In contrast, in their eighth grade year they solved problems that I deemed involved a power-meaning of multiplication.[1] These problems included combinatorics problems and area problems that involved binomial multiplication, quadratic equations, and quadratic functions. The structure of the experiment was based on the conjecture that students' solutions of linear-meaning problems could be a constructive resource in their solution of power-meaning problems—a conjecture that I examine in this paper (RQ2).

This paper focuses on students' solutions of Cartesian product problems like the Outfits and Two-Suit Card Problem.

The Outfits Problem: You have 4 shirts and 3 pairs of pants. An outfit consists of 1 shirt and 1 pair of pants. How many different outfits can you create? (Appendix A, problem 2)
The Two-Suit Card Problem: You have the Ace through King of Hearts—13 cards. Your friend has the Ace through King of Clubs—13 cards. You make two-card hands by pairing one of your cards with one of your friend's cards. How many different two-card hands can you make? (Appendix A, problem 4)

The students solved these kinds of problems during the first five teaching episodes of their eighth grade year (Appendix A lists all problems). I focus on the students' solutions of these problems because they were the first power-meaning problems that were used in the teaching experiment. Given that this was a transition point in the 3-year experiment, I was also interested in responding to the following two research questions:

(RQ3) What were the initial multiplicative concepts that the students used to solve power-meaning multiplication problems?
(RQ4) What differences were there among the students' initial multiplicative concepts to solve power-meaning multiplication problems? Did these differences persist across the year?

Two central results, then, of this paper are: (1) the identification of differences and similarities in the multiplicative concepts that students constructed when solving power-meaning problems (RQ3 and RQ4); and (2) a comparison of the differences and similarities between students' multiplicative concepts when solving linear and power-meaning problems (RQ1).

2. Literature review

In this section, I draw on researchers' mathematical analyses to differentiate between linear- and power-meaning problems. This discussion does not include potentially significant differences in how students reason about problems within

[1] Throughout the paper, I use the terminology "linear-meaning problems" to refer to problems that I deemed involved a linear meaning of multiplication, and "power-meaning problems" to refer to problems that I deemed involved a power meaning of multiplication. I use this terminology with the understanding that this is my classification of the problems not necessarily how the students experienced the problems.