Journal of Mathematical Behavior 32 (2013) 461–473

Contents lists available at ScienceDirect

The Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Covariational reasoning and invariance among coordinate systems

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ARTICLE INFO

Article history:
Available online 5 July 2013

Keywords:
Polar coordinates
Covariational reasoning
Graphing
Function
Multiple representations

ABSTRACT

Researchers continue to emphasize the importance of covariational reasoning in the context of students' function concept, particularly when graphing in the Cartesian coordinate system (CCS). In this article, we extend the body of literature on function by characterizing two pre-service teachers' thinking during a teaching experiment focused on graphing in the polar coordinate system (PCS). We illustrate how the participants engaged in covariational reasoning to make sense of graphing in the PCS and make connections with graphing in the CCS. By foregrounding covariational relationships, the students came to understand graphs in different coordinate systems as representative of the same relationship despite differences in the perceptual shapes of these graphs. In synthesizing the students' activity, we provide remarks on instructional approaches to graphing and how the PCS forms a potential context for promoting covariational reasoning.

1. Introduction

First introduced at the elementary level, graphs are essential representations for the study of numerous mathematical topics including modeling relationships between quantities, exploring characteristics of functions, solving for unknown values, and investigating geometric transformations. Highlighting the central role of graphing in mathematics education, the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices Council of Chief State School Officers, 2010) contains some form of the term *graph* on more than a third of the document's pages. Building from the emphasis on graphing at the K-12 level, graphing is central to the study of several undergraduate mathematics courses: calculus, differential equations, and analysis, to name a few.

Reflecting the heavy focus on graphing in school mathematics, mathematics education research has given significant attention to graphing, with a multitude of studies (e.g., Carlson, 1998; Leinhardt, Zaslavsky, & Stein, 1990; Oehrtman, Carlson, & Thompson, 2008) having investigated students' meanings for graphing in the context of function. Although graphing receives significant attention in mathematics education research, little of this focus has been given to graphing in the *polar coordinate system* (PCS). Complicating the matter, the sparse research (Montiel, Vidakovic, & Kabael, 2008; Montiel, Wilhelmi, Vidakovic, & Elstak, 2009; Sayre & Wittman, 2007) available on students' meanings for the PCS highlights student difficulties, particularly pointing to difficulties involving problematic connections with the *Cartesian coordinate system* (CCS).

In the present study, we explore students' thinking when graphing in the PCS and draw connections with existing research on graphing and function. Specifically, we discuss two undergraduate students' reasoning when graphing in the PCS. To graph relationships in the PCS, both students engaged in several ways of thinking that ranged from determining and plotting

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http://dx.doi.org/10.1016/j.jmathb.2013.05.002

discrete points to reasoning about how quantities continuously vary in tandem. We illustrate these ways of thinking across several different tasks and draw attention to the implications of these ways of thinking relative to the students' ability to flexibly use the PCS and CCS to represent relationships between quantities. Against the backdrop of these findings, we conclude by discussing how the PCS offers a *potential* setting for promoting quantitative and covariational reasoning.

2. Background

Often first introduced in a precalculus course, the PCS is critical for the study of advanced mathematics and can be found in numerous applications in engineering and the sciences. The PCS also plays a central role in exploring complex numbers. For instance, the operations of multiplication, division, and exponentiation are more readily explored when using the polar form of complex numbers. Although the PCS plays an important role in the aforementioned areas, research on student thinking in the context of the PCS is sparse, with a pair of studies by Montiel et al. (2008, 2009) forming the most applicable works to the present study. Both studies included a focus on the PCS, with the earlier study (Montiel et al., 2008) having explored relationships among two-dimensional coordinate systems and the subsequent study (Montiel et al., 2009) having included two- and three-dimensional coordinate systems. The authors made several important observations across these studies including how students' function meanings can cause difficulties when extended to the PCS.

Of relevance to the present study, Montiel et al. (2008) identified that the connections students create between the CCS and PCS are tied to their meanings for function and graphing in the CCS. For instance, students often relied on rules learned in the context of the CCS to determine whether a given relation is a function. These rules included applying the vertical line test to determine whether a graphed relationship in the PCS is a function. Some students also referenced "known" functions when determining whether graphs were functions. By "known," we interpret the authors to mean that the students recalled a shape in the plane that they had previously deemed a function (e.g., a student claiming that a parabola opening down is a function because parabolas are defined as such). Compatible with the earlier study (Montiel et al., 2008), Montiel et al. (2009) noted that when students moved among representational systems, the students' function meanings did not entail coordinating the different conventions of the representational systems, with the students often relying on the conventions from one representational system (e.g., the CCS).

Montiel and colleagues' findings (Montiel et al., 2008, 2009), and specifically students' difficulty in coordinating representational systems, speak to several researchers' (Lobato & Bowers, 2000; Thompson, 1994c, 2013) statements about multiple representations. Lobato and Bowers (2000) questioned, ". . .whether tables, graphs, and equations are multiple representations of *anything* to students" (p. 4). Thompson (1994c) explained:

> Tables, graphs, and expressions might be multiple representations of functions to us, but I have seen no evidence that they are multiple representations of anything to students. In fact, I am now unconvinced that they are multiple representations even to us, but instead may be, as Moschkovich et al. (1993) have said, areas of representational activity among which we have built rich and varied connections. . .I agree with Kaput (1993) that it may be wrongheaded to focus on graphs, expressions, or tables as representations of function, but instead focus on them as representations of something that, from the students' perspective, is *representable*, such as some aspect of a specific situation. The key issue then becomes twofold: (1) to find situations that are sufficiently propitious for engendering multitudes of representational activity and (2) orient students to draw connections among their representational activities in regard to the situation that engendered them. (Thompson, 1994c, pp. 39–40)

Moreover, if students are to conceive multiple representations of *something*, then it is necessary that they not only construct the *something* that is to be represented, but also have meanings for the representational systems such that when the students operate within and move among systems, they can think about their representational activity as conveying the same *something*. Returning to the studies by Montiel and colleagues, the students did not appear to have distinct meanings for the coordinate systems that simultaneously supported connections among the systems. Instead, their meanings were inherently tied to the conventions of one coordinate system (e.g., the vertical line test and the CCS). In short, representations are not external to the person doing the representing but instead consist of a system of mental actions and meanings that the individual has organized into some cognitive structure (von Glasersfeld, 1987).

The construct of covariational reasoning (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Saldanha & Thompson, 1998) foregrounds students' construction of "the something that, from the students' perspective, is *representable*." Covariational reasoning entails the mental actions involved in conceiving two quantities as varying in tandem (Carlson et al., 2002; Saldanha & Thompson, 1998) and is central to students' understanding of numerous secondary mathematics topics including quadratic relationships (Ellis, 2011), exponential relationships (Castillo-Garsow, 2010; Confrey & Smith, 1995), trigonometric relationships (Moore, 2012), rate of change (Carlson et al., 2002; Thompson, 1994a), function (Oehrtman et al., 2008), and the Fundamental Theorem of Calculus (Thompson, 1994b).

In characterizing second-semester calculus students' thinking, Carlson et al. (2002) identified how the students' covariational reasoning influenced their ability to make sense of dynamic situations, interpret graphs, and create graphs. Specifically, the authors identified several mental actions that the students engaged in when coordinating quantities that vary in tandem. These mental actions included, but were not limited to, coordinating directional change (e.g., quantity A increases while quantity B increases), coordinating amounts of change (e.g., the increase in quantity A decreases for successive increases