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Flexible thinking and met-befores: Impact on learning mathematics

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ABSTRACT

In this paper we study the difficulties resulting from changes in meaning of the minus sign, from an operation on numbers, to a sign designating a negative number, to the additive inverse of an algebraic symbol on students in two-year colleges and universities. Analysis of the development of algebra reveals that these successive meanings that the student has met before often become problematic, leading to a fragile knowledge structure that lacks flexibility and leads to confusion and long-term disaffection. The problematic aspects that arise from changes in meaning of the minus sign are identified and the iconic function machine is utilized as a supportive strategy, along with formative assessment to encourage teachers and learners to seek more flexible and effective ways of making sense of increasingly sophisticated mathematics.

1. The role of prior knowledge

Early research into students' difficulties in beginning algebra identified the kinds of errors that were commonly made and investigated the reasons for these errors. The failure to recognize the dual meaning of operation symbols such as the addition and subtraction signs to signal both the action and the result of the action, essential to algebraic understanding, was identified by Wagner (1981) and Kieran (1981). Children's prior learning and early arithmetic experiences (e.g., the belief that multiplication makes larger, that division, like addition is commutative, etc.) were cited as underlying causes of algebraic errors (Matz, 1982). Booth (1984) traced similar errors made by students, aged thirteen to sixteen, which included ideas about the meaning of letters and variables and the use of notation and convention in algebra.

Recent research studies have contributed new insights. MacGregor and Stacey (1997) presented evidence for origins of misinterpretation based on prior learning in students aged 11–15. These included intuitive assumptions and pragmatic reasoning about a new notation, analogies with familiar symbol systems, interference from new learning in mathematics, and the effects of misleading teaching materials.

Vlassis (2002) identified 8th grade students' difficulties and reasoning with negative numbers when the minus symbol preceded or followed terms in a polynomial expression. This revealed an increase in errors when negative whole numbers were introduced to 8th graders after studying algebra with positive numbers for 18 months. Interviews with 18 students revealed the need for a level of flexibility in interpreting the minus sign that the students had not attained that led to erroneous or superficial understanding of algebraic symbolism and operations. For example, students' difficulties in solving an equation such as $-x = 7$ was attributed to a lack of flexibility and failure to recognize that, "the minus sign in this equation

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is the sign attached to the letter *x*, but it is not attached to the value of *x*, which can itself be positive or negative" (Vlassis, 2002, pp. 327–328).

In a subsequent study of twelve 8th graders, Vlassis (2004) reported that all students agreed that the minus sign is used to subtract and no student considered explicitly that the minus sign could have another meaning. This was attributed to the students' earlier experiences with the arithmetic of natural numbers that caused subtle difficulties in manipulating polynomials involving the minus sign. Vlassis introduced the construct of 'negativity' to encompass three different uses of the minus sign in elementary algebra: *unary* applying to a negative number; *binary* applying to subtraction; and *symmetrical* in taking the additive inverse. Student difficulties were related to these differences in meaning, with the need to reconcile prior understandings of arithmetic with algebraic rules when introducing negative numbers.

Despite a large body of research on students' difficulties learning algebra, the teaching of mathematics in American schools leaves many students needing to take remedial courses at college level. This has been the major growth area in two year colleges since 1995 with enrollments exceeding one million for the first time in 2010, constituting more than half the total enrollments in mathematics and statistics (CBMS 2010 Survey, Table, S.2, Table TYE.4). Students who enroll in these courses have either: (1) failed the course previously, at college or in high school; (2) taken the course in high school and passed—but failed to pass a placement exam that qualified them to enroll in a college-level mathematics course; or (3) taken the course several years prior and feel a need to review their skills before taking a college-level mathematics course. Many of these students continue to fail in college and are unable to pursue their chosen career.

The multiple meanings of the minus symbol and the need to interpret symbolism flexibly was identified as a major source of many difficulties for undergraduates, particularly when the symbolism changes meaning in new contexts (McGowen, 1998; McGowen & Tall, 2010). McNeil, Rittle-Johnson, Hattikudur, and Peterson (2010) found that previous experience of solving arithmetic problems hinders the equation-solving performance of undergraduates, even those with high levels of mathematics achievement. Furthermore, experience with a single strategy can reduce problem-solving flexibility and hinder performance solving more complex problems, "not only when the strategy needs to be abandoned and replaced with a novel strategy but also when the strategy is a component skill necessary for carrying out a more complex task" (p. 450).

The need for the learner to cope with subtle changes of mathematical meaning often goes unnoticed in the classroom. Thompson (1994) reminds us of the need to reflect deeply on our own understandings of mathematics and how this affects our teaching and the learning of our students:

> . . . an instructor who fails to understand how his/her students are thinking about a situation will probably speak past their difficulties. Any symbolic talk that assumes students have an image like that of the instructor will not communicate. Students need a different kind of remediation, a remediation that orients them to construct the situation in a mathematically more appropriate way . . .. Whatever students have in mind as they employ symbolic mathematics it often is not the situation their professors intend to capture with their symbolic mathematics (Thompson, 1994, p. 32).

2. Met-befores and flexible thinking

The notion of *met-before* (McGowen & Tall, 2010; Nogueira de Lima & Tall, 2008; Tall, 2004) was introduced to focus on how new learning is affected by the learner's previous experiences. A *met-before* is a mental structure that we have *now* as a result of experiences we have met before. This notion applies both to *supportive* aspects of previous knowledge that form a basis for new learning and to *problematic* aspects that cause confusion and may impede progress. In this paper we emphasize the need not only to determine the causes of problematic aspects, but also to complement this activity with the development of a supportive strategy that seeks to improve the learning experience. Our strategy focuses on the use of the iconic function machine that has a dual interpretation as a process of assignment and a mathematical concept with explicit properties that encourage instructors and students to reflect on the difference between functions such as 'minus' with a unary input and 'subtract' with a binary input (McGowen, DeMarois, & Tall, 2000; Tall, McGowen, & DeMarois, 2000).

2.1. Changes in meaning of the minus symbol

It is widely believed that mathematical notation should be clear and unambiguous. However, the minus sign changes meaning subtly as the curriculum develops. This presents a challenge that some may see as a minor difficulty to be overcome, while others see conflict and increasing disaffection.

Initially the minus sign is encountered in whole number arithmetic as the operation of subtraction, or 'take away.' The minus sign in an expression such as 5–3 means 'start with 5 and take away 3.' The result is 2 and is visibly less than the initial quantity 5. This is consistent with the practical idea that 'take away' leaves less (Fig. 1).

This idea also occurs in Euclidean geometry as 'a common notion', explicitly listed in the foundations of Euclidean geometry in the form 'the whole is greater than the part.'

The second meaning of the minus sign occurs when signed numbers are encountered. Here the symbol −3 refers to the negative number −3 which can be represented as a point on the number line. Now numbers are of three distinct kinds: positive numbers where +3 is the same as the familiar number 3, negative numbers such as −3 and the single number 0 which is neither positive nor negative. While 'subtract 3' is an *operation*, the negative number '−3' is an *object* that can be