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Fifth graders' additive and multiplicative reasoning: Establishing connections across conceptual fields using a graph

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ABSTRACT

This paper looks at 21 fifth grade students as they discuss a linear graph in the Cartesian plane. The problem presented to students depicted a graph showing distance as a function of elapsed time for a person walking at a constant rate of 5 miles/h. The question asked students to consider how many more hours, after having already walked 4 h, would be required to reach 35 miles. To answer this question, the students needed to extend the graph that was presented, either mentally or on paper, as the axes did not go up to 7 h or 35 miles. They also needed to be able to consider not only the total number of hours to reach 35 miles, but also the interval of time after 4 h. The purpose of this paper is to consider the student responses from the viewpoint of multiplicative and additive reasoning, and specifically within Vergnaud's framework of multiplicative and additive conceptual fields and scalar and functional approaches to linear relationships (Vergnaud, 1994). The analysis shows that: some student answers cannot be classified as either scalar or functional; some students combined several kinds of approaches in their explanations; and that the representation of the problem using a graph may have facilitated responses that are different from those typically found when the representation presented is a function table.

1. Introduction

Several decades ago, Vergnaud (1982) defined and described the additive and multiplicative conceptual fields. These definitions and descriptions have proved immensely valuable for the mathematics education community, helping us both to understand the fundamental differences between additive and multiplicative approaches (Carraher, Carraher, & Schliemann, 1985; Kaput & West, 1994; Nunes, Schliemann, & Carraher, 1993; Ricco, 1982; Schliemann, Araujo, Cassundé, Macedo, & Nicéas, 1998; Schliemann & Magalhães, 1990; Schliemann & Carraher, 1992; Schliemann & Nunes, 1990), as well as to design didactical interventions that might foster each kind of approach and thinking (e.g., Schliemann, Carraher, & Brizuela, 2007). However, we have also found that the approaches and ways of thinking are not as dichotomous or in opposition to each other as we once may have thought (e.g., Martinez & Brizuela, 2006; Nunes et al., 1993).

This paper examines the work of 21 fifth grade students as they look at a graph of a linear function in the Cartesian plane (from here on, "graph"). We analyze the student responses to determine how they reason about a function presented to them through a graphical representation. Specifically, this paper addresses the following research questions:

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i. When looking at a graph presented to them in a problem, can students' explanations to questions about the problem be described using the additive and multiplicative framework? Do any student explanations show evidence of thought that does not fit within the existing framework?
ii. How might these explanations be affected by the presence of the graph, as opposed to if a function table had been used?

To address these questions, we will first describe the difference between additive and multiplicative reasoning, as laid out theoretically by Vergnaud (1982). This framework will be used in examining prior research on this topic, as well as analyzing the responses of the students in this study.

We believe that it is important to consider how students use additive and multiplicative relationships, as it is only through an understanding of the multiplicative relationship that students can generalize to a function that is calculable for any value of the independent variable. In addition, studies that complement Vergnaud's framework point to the possible combination of approaches on the part of learners. It is also important to determine how the use of a graph may prompt different reasoning than another representation, such as a function table, might do. Since different representations emphasize different features of the function, as described below, we expect that the use of these representations has a profound effect on student thought (see also Brizuela & Earnest, 2008).

2. Multiplicative and additive reasoning

The primary framework for analyzing this student work is Vergnaud's theory of conceptual fields. In a general sense, a conceptual field is defined as "a set of situations, the mastering of which requires several interconnected concepts. It is at the same time a set of concepts, with different properties, the meaning of which is drawn from this variety of situations" (Vergnaud, 1996, p. 225).

More specifically, Vergnaud (1982) outlines elements of the field of both multiplicative and additive structures. One of the key principles of these fields is that the concepts within them are not acquired quickly, simultaneously, or even within a short period of time; each field may include items which are varied enough in cognitive difficulty that years may separate understanding of different ideas. For example, Vergnaud (1994) defines the multiplicative conceptual field as containing "multiplication and division; linear and bilinear (and *n*-linear) functions; ratio, rate, fraction, and rational numbers; dimensional analysis; linear mapping and linear combinations of magnitudes" (p. 46). In contrast, the additive conceptual field that Vergnaud (1982) describes includes not only addition, subtraction, and the idea of measure, but also "time transformation, comparison relationship, displacement and abscissa on an axis, and natural and directed number" (p. 40).

Using the ideas of the additive and multiplicative fields of understanding, Vergnaud also distinguishes between scalar and functional approaches to relationships between quantities. Simply put, the scalar approach relies on successive addition to model a linear relationship. For example, the scalar method can be applied to the following problem: if it takes 30 min to write 1 page of text, then how long does it take to write 3 pages? Applying a scalar solution, we could reason, "30 min for 1 page, so 60 min for 2 pages, so 90 min for 3 pages." We have done successive additions of the number of minutes required for one page until we reached the number of pages we were seeking. Vergnaud (1994) puts this elegantly in algebraic notation, which would be as follows for our problem:

given: $f(x) = 30x$ where x is the number of pages of text
the scalar solution: $f(3) = 3f(1) = f(1) + f(1) + f(1) = 30 + 30 + 30 = 90$

By contrast, the functional approach rests more directly on the multiplicative relationship between variables. The problem above could be reasoned as "30 min for 1 page, so 3 pages times 30 min per page equals 90 min." Note that this functional approach relies on a single operation on the input variable, rather than successive operations on the output variable. Algebraically, by Vergnaud's (1994) model:

given: $f(x) = 30x$ where x is the number of pages of text
the functional solution: $f(3) = 30(3) = 90$

The functional approach is considered to be more sophisticated (Vergnaud, 1994). As Martinez and Brizuela (2006) point out, it also leads to solutions that are more easily generalizable. The scalar approach necessarily rests on knowing the previous value of the output variable, in order to use the successive addition to find the next value. This, in effect, results in a recursive function. For our example, the function could be considered as:

given: $f(x) = 30x$ where x is the number of pages of text
the recursive solution: $f(3) = f(2) + 30 = [f(1) + 30] + 30 = [30 + 30] + 30 = 90$

However, this does not lead to the establishment of a clear functional relationship between the two different variables that is usable without having determined a previous value. For ease of computation with large values of the input variable, which would require numerous successive additions, or for producing a general functional expression for an unknown value of the input variable, the functional approach becomes important.

The scalar approach also uses or reinforces the belief that there is a countable "next" value for both the independent and dependent variables in a function. While this may, in fact, be the case for many problems with an extra-mathematical context, as students progress in mathematics, they are expected to appreciate the density of the real number line and come to understand the idea of continuous functions. In our example, it would be permissible to consider a case in which two thirds of a page of text had been written, giving $f(2/3) = 30(2/3) = 20$ min. In reviewing how we solved this case, the functional approach was required as there was no "previous" value of the function upon which we could rely. In fact, the