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Counter-examples for refinement of conjectures and proofs in primary school mathematics

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ABSTRACT

The purpose of this study is to explore how primary school students reexamine their conjectures and proofs when they confront counter-examples to the conjectures they have proved. In the case study, a pair of Japanese fifth graders thought that they had proved their primitive conjecture with manipulative objects (that is, they constructed an action proof), and then the author presented a counter-example to them. Confronting the counter-example functioned as a driving force for them to refine their conjectures and proofs. They understood the reason why their conjecture was false through their analysis of its proof and therefore could modify their primitive conjecture. They also identified the part of the proof which was applicable to the counter-example. This identification and their action proof were essential for their invention of a more comprehensive conjecture.

1. Introduction

Proofs and proving are at the heart of mathematics and should be one of the core elements of pupils' mathematical experience from primary school (National Council of Teachers of Mathematics, 2000; Yackel & Hanna, 2003). In the process of conjecturing and proving theorems, mathematicians engage in various activities which are not explicitly described in the final forms of the theorems and proofs. In particular, mathematicians have to continuously test the correctness and coherence of logical chains of their mathematical arguments, for example, by searching for counter-examples. In fact, Lakatos (1976) stated that "informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations" (p. 5), and refutations in which one proposes counter-examples also often play an important role in mathematics.

Pupils' mathematical learning should reflect such mathematical processes to some extent even at primary school level (Lampert, 1990), and proving activities might provide such opportunities for them. In order to conceptualize the notion of proof in school mathematics, Stylianides (2007a) proposed "the intellectual-honesty principle" which "states that the notion of proof in school mathematics should be conceptualized so that it is, at once, honest to mathematics as a discipline and honoring of students as mathematical learners" (p. 3). According to his conceptualization of proof, he further analyzed a classroom episode in which third graders attempted to prove that the sums of two odd numbers were even. As he mentioned that the intellectual-honesty principle "can be useful in conceptualizing other concepts in school mathematics and also more broadly" (Stylianides, 2007a, p. 3), this principle is valuable for not only proofs in school mathematics but also all mathematical learning at all grades.

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The goal of this study is to consider how mathematics teachers and educators could achieve mathematical learning in primary schools that is intellectually honest to mathematical processes with proofs and refutations. The focus of this paper is whether it is possible for primary school children to experience genuine mathematical activities, specifically proofs and refutations, and the author will describe and analyze how two grade 5 children reacted when a counter-example, which is a fundamental method of refutations, was presented in their problem solving process.

2. Theoretical framework

2.1. Literature review and research question

Refutation and counter-examples have played an important role in not only the discipline of mathematics but also mathematical learning. Borasi (1992, 1996) who discussed "an inquiry approach to school mathematics" stated that the uncertainty of mathematics had some positive implications because it could promote continuous inquiry, referring to Lakatos' and Kline's view on the history of mathematics (Kline, 1980; Lakatos, 1976), or more generally, Peirce's notion of inquiry and Dewey's one of reflective thinking. She further conducted a teaching experiment with two high school students, and showed that many valuable learning experiences were promoted by "error activities" which were "instructional activities designed so as to capitalize on the potential of "error" to initiate and support inquiry" (Borasi, 1996, p. 30). She interpreted the term "error" in the most comprehensive way possible, and "even borderline cases such as contradictions, tentative hypotheses and definitions, contrasting results, or results that do not make sense, are considered legitimate starting points for *error activities*" (Borasi, 1996, p. 30, emphasis is original). Therefore, in order to achieve such valuable learning experiences even in primary school mathematics, it is important for mathematics teachers and educators to analyze how pupils refute conjectures or react to counter-examples, because refutation and counter-examples could become starting points for error activities.

Several studies whose participants were secondary school students, undergraduates, student teachers and teachers have already addressed such questions. For example, Hoyles and Küchemann (Hoyles & Küchemann, 2002; Küchemann & Hoyles, 2006) or Peled and Zaslavsky (Peled & Zaslavsky, 1997; Zaslavsky & Peled, 1996) gave a quantitative insight into how secondary school students, student teachers and teachers judged whether statements were true or false. Although participants and tasks were different among these studies, they obtained similar results; for example, not many participants were able to judge that the statements were false with valid justification; some of them justified their judgment by a counter-example, and others explained the general reason why the statements were false.

Other researchers have also explored qualitatively how students treat counter-examples and reexamine their conjectures and proofs. Balacheff (1991) conducted an experiment in which pairs of 13–14 years old students produced ways of calculating the number of diagonals of a polygon. In this experiment, the observer gave counter-examples to the students' way of calculating, and various kinds of students' approaches to overcome contradiction by counter-examples were identified: for example, modification of conjectures or counter-examples considered as exceptions. Balacheff indicated the factors which appeared to determine their approaches (the problem itself, a global conception of what mathematics consists of, the situation, and the type of conjecture) and concluded that, in order to base the learning of mathematics on students' awareness of a contradiction according to a constructivist approach, mathematics educators had to consider that students' approaches to overcome the contraction were quite uncertain.

Zazkis and Chernoff (2008) introduced the notion of "pivotal example" and "bridging example", and analyzed an interview with a female prospective primary school teacher. In the interview, during her attempt to reduce a fraction, she initially conjectured that the products of two prime numbers were prime numbers. When interviewer required her to think about the number 15, she acknowledged that 15 was a product of two primes and was not a prime, but she seemed to maintain her initial conjecture. After that, the interviewer asked her to consider the number 77, and she then abandoned her initial conjecture for the first time. From this interview, Zazkis and Chernoff concluded that while one counter-example was mathematically enough to refute a statement, different examples influenced differently the extent to which learners were convinced of the falseness of the statement.

According to the methods of mathematical discovery which Lakatos (1976) had illustrated, Larsen and Zandieh (2008) constructed a framework for research into learning and teaching mathematics. They classified types of activity into monster-barring, exception-barring and proof-analysis, and indicated focuses and outcomes of each activity. They further described the process in which undergraduates refined their conjectures regarding groups and subgroups in abstract algebra through proofs and refutations, and suggested that the framework was a useful tool for not only making sense of classroom mathematical activity but also designing instructional approaches that supported the guided reinvention of mathematics.

However, there have been few studies so far which have investigated how primary school students react to counter-examples. One such example is the study by Reid (2002), who described, using the terms of Lakatos (1976), the process in which small groups of fifth graders engaged in conjectures and refutations on the chessboard problem. They were asked to find all the squares in some particular grids (for example, 4 by 4 and 5 by 5), and made conjectures on the number of squares in an n by n grid. In the process, they made their conjectures from their observation of patterns and tested the validity of their conjectures empirically; and after they found counter-examples of their conjectures, they returned to pattern observation or excluded the counter-examples as 'monsters' or exceptions.

Reid (2002) showed that primary school students also reacted to counter-examples diversely, but pupils in his study did not prove their conjectures but tested them only empirically, as a part of the paper's title "conjectures and refuta-