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Should proof be minimal? Ms T's evaluation of secondary school students' proofs☆

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ABSTRACT

Calls for reform in mathematics education around the world state that proofs should be part of school mathematics at all levels. Turning these calls into a reality falls on teachers' shoulders. This paper focuses on one secondary school teacher's reactions to students' suggested proofs and justifications in elementary number theory. To determine whether the justifications are acceptable, the teacher used not only her SMK regarding mathematical aspects, but also her PCK about what a student giving this justification might know or not know. A discussion of the findings is followed by some questions that arise.

1. Prologue

Tom, a middle school student, was asked: Is the following statement true: "The sum of any four consecutive natural numbers is divisible by four"?

Tom provided the following answer: "The statement is false. I checked the sum of the first four consecutive numbers: $1+2+3+4=10$, and 10 is not divisible by 4. The sum of the next four consecutive numbers is obtained by adding 4 to this sum (each of the four numbers in the sum grows by 1, so the sum grows by 4). It is known that adding 4 to a sum that is not divisible by 4 will yield a sum that is not divisible by 4 either. And so on, each time we add 4 to a sum that is not divisible by 4, and thus obtain another sum that is not divisible by 4. Therefore, the statement is false."

Let us assume that you are Tom's teacher. What is your opinion about Tom's answer? How would you evaluate his answer? Would you accept it as a valid justification which refutes the statement? On what grounds?

In this paper we follow one teacher's evaluation of students' justifications of elementary number theory (ENT) statements, and her explanations of her related decision-making processes.

2. Introduction

"Proofs, I maintain, are the heart of mathematics" (Rav, 1999, p. 6). As such, proof is a key component of any mathematical education. According to Bell (1976), the mathematical meaning of proof carries three aspects, each important on its own right:

> The first is *verification or justification*, concerned with the truth of a proposition; the second is *illumination*, in that a good proof is expected to convey an insight into *why* the proposition is true; this does not affect the *validity* of proof,

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> but its presence in a proof is aesthetically pleasing. The third sense of proof is most characteristically mathematical, that of *systematization*, i.e. the organization of results into a deductive system of axioms, major concepts and theorems, and minor results derived from these (p. 24).

Bell's three characteristics are used as a tool for examining proofs within the mathematics education community. Davis, Hersh and Marchisotto (1995) mentioned some additional aspects of proofs that are commonly accepted, and may be used in various didactical situations:

> In being exposed to the scrutiny and judgment of a new audience, the proof is subject to a constant process of criticism and revalidation. Errors, ambiguities, and misunderstandings are cleared up by constant exposure. . . . Proof, in best instances, increases understandings by revealing the heart of the matter. . . . Finally, proof is ritual, and a celebration of the power of pure reasoning (p. 167).

That is, being engaged in the act of proving may help students better *understand* mathematics, and *admire* its power.

Thus, it is only natural that calls for reform in school mathematics across the world recommend broadening the variety of mathematical statements whose proofs are addressed in mathematics classrooms. For example, the National Council of Teachers of Mathematics states that:

> Mathematical reasoning and proof offer powerful ways of developing and expressing insights about a wide range of phenomena. Reasoning and proof should be a consistent part of students' mathematical experiences in prekindergarten through grade 12 (NCTM, 2000, p. 56).

Similar recommendations can be found in documents of the Australian Education Council (1991), and of the Israeli Ministry of Education (e.g., 1994).

Stylianides (2007) suggested a definition for the act of proving that is embedded in the classroom context:

Proof is mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justifications;
2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community (p. 107).

Stylianides emphasizes not only the content of proof (1), but gives equal significance to modes of argumentation (2) and to the ways the arguments are represented (3). In this paper, we will refer to the latter two components as the *proof framework*.

One may wonder how these recommendations can become part of mathematics learning in a classroom. The mathematics teacher can enact the requirement to broaden students' access to proofs in a classroom, since "students learn mathematics through the experiences that teachers provide. Thus, students' understanding of mathematics, their ability to use it . . . are all shaped by the teaching they encounter in school" (NCTM, 2000, p. 16). There is a widespread agreement that teachers have a crucial impact on their students' mathematical knowledge (e.g., Hart Research Associates, 2005; Presley & Gong, 2005).

It is widely accepted that "[t]eachers' subject matter conceptions have a significant impact on their instructional practices" (Knuth, 2002a, p. 63). That is, teachers' performance is influenced by their subject matter knowledge (SMK), which refers to "the amount and organization of knowledge per se in the mind of the teacher" (Shulman, 1986, p. 9).

SMK, in the case of proving, entails a teacher's ability to prove or refute various types of statements. ENT is one of the recommended domains for incorporating proving into classroom practice: "Middle grade students can develop justifications to support their conclusions in varied topics, such as number theory" (NCTM, 2000, p. 264). ENT provides opportunities to pose simple yet non-trivial challenging problems accessible to middle school and high school students. Teachers need to be familiar with ways and methods of rationally examining, validating or refuting mathematical statements. What does research tell us about prospective and practicing teachers' SMK on these issues? Some research has been done with respect to middle and high school teachers' and prospective teachers' knowledge on proving universal and existential statements. Dreyfus (2000), following work with high school students by Healy and Hoyles' (1998), presented 44 secondary school teachers with 9 justifications to the universal claim, "The sum of any two even numbers is even." He found that most secondary school teachers easily recognize formal proofs, but have little or no appreciation for other types of justifications such as verbal, visual or generic. Dreyfus referred to proof which showed a general argument as it applies to a numerical example as generic. The terms verbal or visual were used by Dreyfus as descriptors for particular proofs used by Healy and Hoyles.

Knuth (2002b) examined in-service secondary school mathematics teachers' conceptions of proof. His findings suggest that teachers recognize the variety of roles that proof plays in mathematics. Noticeably absent, however, was a view of proof as a tool for learning mathematics. Many of the teachers held limited views of the nature of proof in mathematics and demonstrated inadequate understandings of what constitutes proof. He found that: "In determining the justification's validity, these teachers seemed to focus solely on the correctness of the algebraic manipulations rather than on the mathematical validity of the justification" (p. 392).