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Undergraduate mathematics majors' writing performance producing proofs and counterexamples about continuous functions

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ABSTRACT

In advanced mathematical thinking, proving and refuting are crucial abilities to demonstrate whether and why a proposition is true or false. Learning proofs and counterexamples within the domain of continuous functions is important because students encounter continuous functions in many mathematics courses. Recently, a growing number of studies have provided evidence that students have difficulty with mathematical proofs. Few of these research studies, however, have focused on undergraduates' abilities to produce proofs and counterexamples in the domain of continuous functions. The goal of this study is to contribute to research on student productions of proofs and counterexamples and to identify their abilities and mathematical understandings. The findings suggest more attention should be paid to teaching and learning proofs and counterexamples, as participants showed difficulty in writing these statements. More importantly, the analysis provides insight into the design of curriculum and instruction that may improve undergraduates' learning in advanced mathematics courses.

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1. Introduction

Proving and refuting are crucial abilities in advanced mathematical thinking because they help demonstrate whether and why propositions are true or false. In the mathematics community, proving and refuting are inextricably linked given the role each plays in establishing mathematical knowledge (Lakatos, 1976). A mathematical proof requires that definitions, statements, or procedures are used to "deduce the truth of one statement from another" (Tall, 1989, p. 30), helping people understand the logic behind a statement and "insight into how and why it works" (Tall, 1992, p. 506). Counterexamples similarly play a significant role in mathematics by illustrating why a mathematical proposition is false; a single counterexample is sufficient to refute the falsity of statements (Peled & Zaslavsky, 1997). Taken together, mathematical proofs and counterexamples can provide students with insight into meanings behind statements and also help them see why statements are true or false. Accordingly, undergraduate students in advanced mathematics are expected to learn and to use both proofs and counterexamples throughout the undergraduate mathematics curriculum.

Before constructing a proof for a true statement or generating a counterexample for a false one, students and teachers need to be able to accurately decide the truth or falsity of a given proposition. Research investigating undergraduate students' and mathematics teachers' ability to evaluate a given proposition, however, suggest that many of them have difficulty verifying the truth and falsehood of given statements due to their inadequate understanding of the mathematical content (Barkai, Tsamir, Tirosh, & Dreyfus, 2002; Riley, 2003). Despite the importance of teaching and learning proofs and counterexamples, Thurston

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(1994) found that mathematicians often struggle with explaining how to write formally complete mathematical proofs to students and concluded that more effective ways of teaching these are needed. Current research also supports Thurston's conclusions: studies have shown that many undergraduates and mathematics teachers who have completed several advanced mathematics courses at the collegiate level or who have received a bachelor's or master's degree in mathematics still have difficulty with proofs (e.g., Cusi & Malara, 2007; Goetting, 1995; Harel & Sowder, 1998; Knuth, 1999, 2002a, 2002b; Martin & Harel, 1989; Moore, 1990, 1994; Morris, 2002; Stylianides, Stylianides, & Philippou, 2004, 2007; Weber, 2001, 2004) and counterexamples (e.g., Barkai et al., 2002; Peled & Zaslavsky, 1997; Zaslavsky & Peled, 1996). When facing mathematical proofs, many undergraduate students seem to lack adequate understandings of the components of mathematical proofs (e.g., Harel & Sowder, 1998; Martin & Harel, 1989), or have insufficient conceptual understandings of writing mathematical proofs (e.g., Moore, 1990, 1994; Weber, 2001, 2004).

Research also has documented undergraduates' difficulties coordinating their informal and formal understandings of the concept about continuous functions (e.g., Bezuidenhout, 2001; Ferrini-Mundy & Graham, 1994; Lauten, Graham, & Ferrini-Mundy, 1994; Shipley, 1999; Tall & Vinner, 1981; Vinner, 1992; Wilson, 1994; Williams, 1991), which is essential content across the world in college mathematics (as well as in pre-calculus and calculus courses in high school). Findings indicate that college students have difficulty connecting the ideas of continuity and functions (Lauten et al., 1994; Vinner, 1992; Wilson, 1994), use their partially correct concept image – defined as mental pictures – to think about continuity[1] (Ferrini-Mundy & Graham, 1994; Tall & Vinner, 1981), and possess inadequate understandings between continuity and limits (Bezuidenhout, 2001; Williams, 1991). Even though undergraduates currently learn an important theorem in the domain of continuous Functions – the Intermediate Value Theorem – in a class with a focus on writing mathematical proofs, some of them are still unable to provide a valid proof for the Intermediate Value Theorem because they do not understand the proof for that theorem (Shipley, 1999).

Yet despite the importance of proofs and counterexamples in undergraduate mathematics and the difficulty students have producing and comprehending proofs and counterexamples, few studies have focused specifically on students' abilities to produce proofs and counterexamples in the domain of continuous functions—a domain that is both central to and pervasive in undergraduate mathematics and that students have learned in their previous calculus course and currently learn in their advanced calculus. The main purpose of this study is to examine undergraduate mathematics majors' performance constructing proofs and generating counterexamples This study was guided by two research questions: (1) How well do undergraduates construct proofs and generate counterexamples in the domain of continuous functions? (2) What problems appear in the proofs students construct or the counterexamples they generate? We hypothesized that the majority of participants were able to evaluate given propositions correctly as well as to produce complete proofs and counterexamples about continuous functions since they all had studied the topic. The results reported in this article focused on undergraduate mathematics majors' written responses regarding proofs and counterexamples about continuous functions.

2. Conceptions of proof

From a traditional perspective, "a mathematical proof is a formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion" (Griffiths, 2000, p. 2). Stylianides (2007) defined proof to include the essential components of *sets of accepted statements*, *modes of argumentation*, and *modes of argument representation*. In this definition, proof serves as a means to communicate thoughts with learners in the mathematics community. Common sense suggests that individuals who understand what constitutes a mathematical proof may be more successful at evaluating purported arguments or their written responses as a valid proof or not. Indeed, "[a] person's (or a community's) proof scheme consists of what constitutes ascertaining and persuading for that person (or community)" (Harel & Sowder, 2007, p. 809); therefore, Harel and Sowder (1998) provided proof schemes in order to identify students' individual proof work which they are convinced.

In order to better characterize undergraduates' proof productions for a true proposition, three proof classifications from Harel and Sowder's (1998) framework applied to the study reported here. The first proof category, the inductive proof scheme, describes how individuals convince themselves or persuade others by providing one or more particular examples, which corresponds to Balacheff's (1988) naive empiricism (verification by several randomly selected cases) and crucial experiment (verification by carefully selected cases). Finlow-Bates, Lerman, and Morgan (1993) and Healy and Hoyles (2000) used a similar term, empirical, to indicate that students produce examples as proofs to convince themselves.

The second proof category, non-referential symbolic proof scheme, demonstrates that individuals employ symbolic manipulations with little or no coherent understandings of their meanings. In other words, students manipulate symbols with no "functional or quantitative reference[s]" (Harel & Sowder, 1998, p. 250). The third proof category, the structural proof scheme, suggests that individuals realize that "definitions and theorems belong in the structure created by a particular set of axioms" (Knapp, 2006, p. 28). This proof scheme is similar to Balacheff's (1988) calculations on statements and Weber's (2004) and Weber and Alcock's (2004) syntactic proof productions. According to Balacheff (1998), calculations on statements mean students rely on definitions, theorems, or explicit properties related to the statement when producing a proof. Similarly, Weber (2004) and Weber and Alcock (2004) described how an individual attempts to construct a proof by stating the

[1] Continuity in this paper refers to the continuity of functions.