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Does mathematical learning occur in going from concrete to abstract or in going from abstract to concrete?

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Abstract

The notions of *abstract* and *concrete* are central to the conceptualization of mathematical knowing and learning. It is generally accepted that development goes from concrete toward the abstract; but dialectical theorists maintain just the opposite: development consists of an ascension from the abstract to the concrete. In this article, we reformulate the relationship of abstract and concrete consistent with a dialectical materialist approach to conscious human activity, as it was developed in the line of cultural-historical psychology. Our reformulation of development in and through interpretation shows that rather than being a movement from concrete to abstract or from abstract to concrete, development occurs in a double ascension that simultaneously moves in both direction: it is a passage of one in the other. In the proposed approach, the theoretical contradictions of earlier approaches to the issue of abstract have been eliminated.

Keywords: Abstract; Concrete; Generalization; Dialectical logic; Double ascension

> Who thinks abstractly? The uneducated person, not the educated. (Hegel, 1988, p. 575)[1]

> It is necessary to challenge the ideas of proof, *abstraction*, calculation, low level, and so forth. (Walkerdine, 1997, p. 58, our emphasis)

Who thinks abstractly? Without doubt, most mathematics educators would answer something like “the educated” or “those who have developed formal operations (reasoning).” It is evident that their responses are precisely the opposite of what Georg W.F. Hegel articulated in our introductory quote. This difference became salient to us during an expert/expert study where experienced and highly successful scientists had been asked to interpret graphs from undergraduate courses in their own discipline (Roth & Bowen, 2003). It turned out that in some instances, half of them did not arrive at the answers that the professor teaching the course from which the graphs had been culled would have accepted as correct. Our scientist participants struggled making sense of the graphs, taken to be abstract representations of natural phenomena (Latour, 1993). However, those scientists who succeeded articulated a lot of concrete detail and used their lived experiences during their interpretations. To better understand the relation between the abstract (abstraction) and concrete (concretion), we constructed a detailed case study of one scientist (Eddie)

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[1] We used several non-English works. All translations are ours.

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interpreting a graph featuring the distributions of three types of plants; the graph is used in ecology courses in support of a theory of adaptation (see Roth & Hwang, 2006).

Our case study articulates a trajectory in the course of which Eddie moved from not knowing the sense of the graph to the production of a statement that "correctly" described the graph in terms of an abstraction: Differential adaptation to climate leads to differences in the distribution of plants with different photosynthetic mechanisms. The trajectory constitutes an episode of learning, where Eddie articulated and appropriated the sense of this graph. Interestingly, our case study showed how the movement along this trajectory simultaneously is from concrete to abstract and from abstract to concrete. In our paper we point out that this is not a case of two compatible movements, the first from concrete to abstract and the second in the opposite direction. Rather, the learning trajectory is equivalent to a double ascension (some authors use the term *ascent* instead)—the simultaneous movement from abstract to concrete and concrete to abstract. This double ascension is precisely the movement along which sense comes to be articulated; and "sense is the ideality of the sensible and the sensibility of the idea: it is the passage of the one in the other" (Nancy, 2002, p. 49). The idea (abstract, general) and the sensible world (particular, concrete) no longer are separate—one is the passage *in* (not into!) the other. "Sense passes between the two, from the one to the other absence of sense, from the one to the other truth" (p. 50).

The purpose of this article is to develop and ground a dialectical theory in which mathematical learning is described as a double ascension the simultaneously and contradictorily moves from abstract to concrete and from concrete to abstract. In the process, the dichotomy apparent in our title comes to be negated and sublated (a term that has the sense of making cease, integrate, and being overcome).

1. The problematic

The concept of abstraction, which has recently received a lot of attention from mathematics educators (Dreyfus & Gray, 2002), really has been an important topic of philosophical and mathematical thinking since the times of the ancient Greek. *Abstraction* literally means drawing (Lat. *trahere*) away (Lat. *ab[s]*) and frequently is associated with the idea that it is characteristic of intelligence and higher-order cognition (Ohlsson & Lehtinen, 1997). Naturalist philosophers define abstraction (or generalization) as a process, where human beings move(d) from simple observation sentences or terms (e.g., ∃ x(x is raven)) to focal observation categoricals (e.g., ∃ x(x is raven. ⊃ x is black)) to observation *categoricals* that describe a number of observations or classes of objects (e.g., ∀ x(x is raven ⊃ x is black) (Lakoff, 1987; Quine, 1995; Russell, 1938; Wittgenstein, 1978). Not everyone agrees, though, with this characterization. There are suggestions that already in the observation of a new object, the object involves a unity of appearances such that the apperception of the object *as* object is "in general the first universal typification—precisely the [typification of the object] as an object of experience, an object of perception, and the [typification of unities] as a configuration of objects" (Husserl, 1945, p. 335). In any event, abstraction comes off as one of the highest human achievements, whereas concreteness is, as in Piaget's stage theory, characteristic of lower levels of thinking. However, the concept of abstraction is not without ambiguity and, as our first introductory quote shows, there are contrary formulations that attribute abstract thinking to the undeveloped person (mind).

A different way of thinking about abstract and concrete suggests that concreteness "is that property which measures the degree of our relatedness to the object (the richness of our representations, interactions, connections with the object), how close we are to it, or, if you will, the quality of our relationship with the object" (Wilenski, 1991, p. 198), where, in our interpretation, the relationship may be to a material object or to an idea. This statement therefore also allows us to understand the opening Hegel quote in a new way, as "uneducated" may be interpreted to mean that a person has less rich representations, interactions, and connections with the (material, ideal) object of activity. In fact, dialectical philosophers generally describe development of scientific cognition to occur in a process of ascending from abstract to concrete (e.g., Il'enkov, 1982).

The two ways of thinking about the terms *abstract* and *concrete* lead us to a contradiction: learning and development appear to simultaneously constitute a movement from concrete to abstract *and* a movement from abstract to concrete. In other words, it is not two concurrent movements one going from abstract to concrete and the other from concrete to abstract. Rather, it is the same movement (development, learning) that simultaneously goes from abstract to concrete and from concrete to abstract. To put it in logical terms: if p is the statement "movement is from concrete to abstract," then $\neg p$ stands for its negation, "movement is from abstract to concrete." What our statement therefore suggests is that $p = \neg p$. This clearly constitutes a contradiction and is not permissible in classical logic—a statement p and its negation