Learning and Instruction 37 (2015) 1–4

Contents lists available at ScienceDirect

Learning and Instruction

journal homepage: www.elsevier.com/locate/learninstruc

Unraveling the gap between natural and rational numbers

Keywords:
Rational numbers
Conceptual change
Dual process theory

A B S T R A C T

The foundations for more advanced mathematics involve a good sense of rational numbers. However, research in cognitive psychology and mathematics education has repeatedly shown that children and even adults struggle with understanding different aspects of rational numbers. One frequently raised explanation for these difficulties relates to the natural number bias, i.e., the tendency to inappropriately apply natural number properties to rational number tasks. This contribution reviews the four main areas where systematic errors due to the natural number bias can be found, i.e., their size, operations, representations and density. Next, we discuss the major theoretical frameworks from which rational number understanding is currently investigated. Finally, an overview of the various papers is provided.

Number sense is a central theme in current cognitive and developmental (neuro)psychology, educational psychology and (psychology of) mathematics education (e.g., Kaufmann & Dowker, 2009). Typically, cognitive (neuro)psychologists (e.g., Ansari & Karmiloff-Smith, 2002; Dehaene, 1997) consider number sense as the rapid and accurate perception of small numerosities and the ability to compare numerical magnitudes, and to comprehend simple arithmetic operations. A central finding is that infants and young children are already able to understand and manipulate numerical magnitude information using non-symbolic magnitude representations (e.g., Xu & Spelke, 2000), which later also applies to magnitudes represented symbolically (Sekuler & Mierkiewicz, 1977, however, for a criticism see Negen & Sarnecka, in press). In some studies, performance on these very basic symbolic and non-symbolic comparison and estimation tasks has been found associated with individual differences in later general mathematics achievement in general (e.g., De Smedt, Verschaffel, & Ghesquière, 2009; Halberda, Mazzocco, & Feigenson, 2008) and in specific aspects of mathematics achievement, such as mental computation, in particular (Linsen, Verschaffel, Reynvoet, & De Smedt, 2015).

However, this number sense research typically has focused on the recognition and manipulation of numerosities, and thus on *natural* numbers, and its relationship with mathematics achievement. However, it is widely acknowledged that one of the foundations for more advanced mathematics, including algebra and probability, involves a good sense of rational numbers (Clarke & Roche, 2009; Lamon, 2005). For example, Siegler et al. (2012) found that fifth graders' fraction knowledge predicted algebra and overall mathematics scores in high school, even after they had controlled for various other variables such as reading achievement, IQ, working memory, whole number knowledge, family income, and family education. Still, research in cognitive psychology and mathematics education has repeatedly shown that children and even adults struggle with understanding different aspects of rational numbers, and especially fractions (Behr, Wachsmuth, Post, & Lesh, 1984; Cramer, Post, & delMas, 2002; Mazzocco & Devlin, 2008; Merenluoto & Lehtinen, 2002; Vamvakoussi, Christou, Mertens, & Van Dooren, 2011; Vamvakoussi, Van Dooren, & Verschaffel, 2012). Even many student-teachers and teachersstruggle with the understanding of rational numbers (Clarke & Roche, 2009; Merenluoto & Lehtinen, 2004; Post, Cramer, Behr, Lesh, & Harel, 1993).

While learners' difficulties with understanding rational numbers has been found to have several possible sources (Siegler, Fazio, Bailey, & Zhou, 2013), an active field of research focused on one particular explanation for these difficulties, namely the natural number bias (e.g., De Wolf & Vosniadou, 2011; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013; Vamvakoussi et al., 2011, 2012; Vamvakoussi & Vosniadou, 2004; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013). The natural number bias refers to the idea that difficulties with rational numbers may arise from the inappropriate application of natural number properties. Indeed, while a good natural number sense seems crucial in the development of mathematical understanding, this same natural number understanding may also *interfere* in mathematical reasoning when rational numbers are involved, in the sense that learners implicitly or explicitly assume that the features of natural numbers continue to apply to rational numbers. This causes systematic errors to arise when rational numbers behave differently from natural numbers (Lamon, 1999; Moss, 2005; Resnick et al., 1989; Smith, Solomon, & Carey, 2005; Vamvakoussi & Vosniadou, 2010), while rational number tasks that are compatible with natural number thinking do not elicit such errors (e.g., Nunes & Bryant, 2008). Because learners' reliance on natural number reasoning when dealing with rational number tasks appears to have these particular characteristics—facilitation of reasoning when it is appropriate to use one's natural number knowledge, and the adverse effect

http://dx.doi.org/10.1016/j.learninstruc.2015.01.001
0959-4752/

when it is not—the term *bias*, first introduced by Ni and Zhou (2005) in relation to this phenomenon, seems justified.

While there is no consensus about the origins of the natural number bias, it is obvious that in the first years of a child's life intuitions about natural numbers are much more often externalized and systematized in social interaction than any intuitions about rational numbers (Greer, 2004). Indeed, the natural number concept is mediated from early on in children's development by language and finger counting. Also early mathematics instruction focuses on natural number knowledge and arithmetic, thus supporting the systematization and validation of children's initial understanding of number as natural. In sum, long before children are introduced to rational numbers at school, they have already constructed a rich, extended and persistent understanding of number grounded in knowledge of natural numbers (Gelman, 2000; Smith et al., 2005; Vamvakoussi & Vosniadou, 2010). On the other hand many studies have found that young children display a variety of skills for dealing with quantitative relations and proportional reasoning before any formal teaching of rational numbers (Frydman & Bryant, 1988; McMullen, Hannula-Sormunen, & Lehtinen, 2013, 2014; Singer-Freeman & Goswami, 2001; Sophian, 2000). However learners might have difficulties to integrate this knowledge with the number concept on a more abstract level (see Haider et al., 2014).

When rational numbers are introduced in the curriculum, typically in the middle years of elementary school, several expectations about the features and behavior of numbers are violated. It is exactly in situations where this happens that systematic errors in learners' mathematical reasoning due to the natural number bias occur. The research literature distinguishes four main areas where such systematic errors can be found.

First, determining the size of a rational number is based on different principles than for a natural number. For instance, when comparing two fractions the counting sequence, which applies to natural numbers (1, 2, 3…), is no longer useful. The size of a fraction can also not be determined by considering only the numerator or denominator, yet a mistake made by many learners is to assume that when a fraction's denominator, numerator, or both increase, the numerical value of the total fraction increases (Mamede, Nunes, & Bryant, 2005; Meert, Grégoire, & Noël, 2010). Also, contrary to natural numbers, the length of a number does not longer help to decide which number is larger, but learners are reported to think that “longer decimals are larger” and “shorter decimals are smaller” (Resnick et al., 1989).

Second, arithmetic operations with rational numbers can lead to unexpected outcomes if one thinks in natural number terms. During the first years of elementary school, children have only done multiplications and additions that always led to a result that was larger than the operand, and only divisions and subtractions that led to a smaller result. It has been repeatedly found that learners apply these expectations also to operations with rational numbers, for instance incorrectly assuming that 3 multiplied by 0.71 will result in an outcome larger than 3 (Hasemann, 1981; Vamvakoussi et al., 2012).

Third, while natural numbers have only one symbolic representation, rational numbers can be represented in different ways (i.e., by fractions as well as decimals), and within each of these two major representational types even by an infinite number of possible representations (e.g., 0.75, 0.750, 3/4, 75/100, …). Research has indeed shown that learners often do not see fractions and decimals as representations of the same number (Vamvakoussi et al., 2012) and moreover consider a fraction as two (natural) numbers instead of as a number in its own right (e.g., Smith et al., 2005).

Fourth, whereas natural numbers are discrete (i.e., one can say which number follows a given number), rational numbers are dense (i.e., there are infinitely many numbers between any two given rational numbers, and there is no such thing as a number following another one). Many studies have shown that learners have difficulties to understand that between two pseudo-successive numbers such as 0.2 and 0.3 there are infinitely many numbers (Merenluoto & Lehtinen, 2004; Vamvakoussi et al., 2011).

Given that a lot of the systematic errors learners make in rational number tasks can be explained by the inappropriate application of natural number principles, research in this area often departs from the theory of conceptual change (Vosniadou, Vamvakoussi, & Skopeliti, 2008), which assumes that a major source of difficulty in conceptual understanding and learning is the incompatibility between (largely unconscious) background assumptions of learners and scientific ideas that are addressed in instruction. In the last decade various research groups have started to apply these insights also to mathematics learning (see an earlier special issue of *Learning and Instruction*, Vosniadou & Verschaffel, 2004). As will become clear in the current special issue, the conceptual change perspective shows to be particularly fruitful to understand the gap between natural and rational number understanding, and the errors made by learners.

More recently, the focus of research on rational number understanding has included adults who do not longer necessarily commit systematic errors in rational number tasks. The main idea behind this kind of research is that despite the fact that educated adults have gained a correct understanding, they may still be affected somehow by the natural number bias, but may be more successful in overcoming it, which is why they no longer commit errors. This explanation is based on the dual-process theory of reasoning (Epstein, 1994; Evans & Over, 1996; Kahneman, 2000; Sloman, 1996; Stanovich, 1999), which differentiates between intuitive and analytic types of reasoning (a distinction that is prominent also in the field of mathematics education, for an overview, see Gillard, Van Dooren, Schaeken, & Verschaffel, 2009; Inglis & Simpson, 2004; Leron & Hazzan, 2006). According to dual process theories of reasoning, by default one tends to use very fast reasoning processes called intuitive or heuristic processes when interpreting a situation or task, and only in some cases one will also employ slower and more effortful analytic reasoning processes. In this line of work, it is argued that even after someone has accomplished a conceptual change in the domain, a correct understanding of rational numbers may coexist with an earlier, more primitive understanding of natural number which is intuitive in nature. When solving a rational number task, both may come into play and concur with each other. In some cases, intuitive answers may be produced (which explains that systematic errors occur on some rational number tasks). In other cases, correct answers may be given because the intuitive, natural number based reasoning is successfully overcome. In the latter cases, however, responding correctly will need more time than in problems where the correct answer is in line with natural number knowledge. Previous studies (Obersteiner et al., 2013; Vamvakoussi et al., 2012, 2013) have successfully shown that the reaction times of educated adults and even mathematical experts still show signs of a natural number bias on a variety of rational number tasks.

The contributions to the special issue present state-of-the-art empirical research conducted across the whole range of manifestations of the natural number bias, using a wide variety of tasks (comparing, ordering or calculating with algebraic expressions, fractions, decimals), methodologies (new statistical techniques, reaction time research), and age groups (from 9-year olds to adults). They point to new findings and increased insights in underlying mechanisms and potential explanations, but also toward educational implications of the obtained findings and insights. All contributions focus on the rational number concept, but their theoretical,