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Modeling the developmental trajectories of rational number concept(s)

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ABSTRACT

The present study focuses on the development of two sub-concepts necessary for a complete mathematical understanding of rational numbers, a) representations of the magnitudes of rational numbers and b) the density of rational numbers. While difficulties with rational number concepts have been seen in students' of all ages, including educated adults, little is known about the developmental trajectories of the separate sub-concepts. We measured 10- to 12-year-old students' conceptual knowledge of rational numbers at three time points over a one-year period and estimated models of their conceptual knowledge using latent variable mixture models. Knowledge of magnitude representations is necessary, but not sufficient, for knowledge of density concepts. A Latent Transition Analysis indicated that few students displayed sustained understanding of rational numbers, particularly concepts of density. Results confirm difficulties with rational number conceptual change and suggest that latent variable mixture models can be useful in documenting these processes.

1. Introduction

Students have serious difficulties with learning about fractions and decimals, and many educated adults do not have an adequate conceptual understanding of rational numbers (Merenluoto & Lehtinen, 2004; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Van Hoof, Vandewalle, Verschaffel, & Van Dooren, in this issue; Vosniadou & Verschaffel, 2004). Despite having many features that can be generalized from natural numbers (Steffe & Olive, 2010; Torbeyns, Schneider, Xin, & Siegler, in this issue), learning and understanding rational number concepts has proven to be challenging for most learners. One issue with learning about rational numbers is that there are many features of natural numbers that cannot be extended to rational numbers. Thus, learning about rational numbers is not a matter of simply developing a deeper or broader understanding of natural numbers. Understanding rational numbers requires significant change in reasoning about number concepts, for example: A rational number with smaller integers can represent a larger magnitude (e.g. in the denominator of a fraction); there are always an infinite number of numbers between any two numbers; one cannot say what is the next number in a sequence of rational numbers. All of these concepts of rational numbers clash with students' previous understandings of the nature of numbers. Thus, deep and significant conceptual change is required in order to fully grasp the nature of rational numbers, and this change is hard to come by.

The present study tracks, over a one-year period, the development of conceptual knowledge of third to fifth grade children. We aim to identify key stages in the development towards a coherent and mathematically correct conceptual understanding of rational numbers in students who are first learning about rational numbers. In doing so, we hope that what emerges is a more fully developed understanding of rational number conceptual change processes. In particular, we aim to uncover how the different sub-concepts of rational numbers develop.

1.1. Rational number conceptual change

Conceptual change with rational numbers is a complex phenomenon, which can be operationalized with different sub-concepts. In this study, we focus on concepts related to the representations of the magnitudes of fractions and/or decimals and the density of fractions or decimals. Learning both of these sub-concepts violates the previous concepts students have of numbers, based on their natural number knowledge (DeWolf & Vosniadou, in this issue; Vamvakoussi, Christou, Mertens, & Van

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Dooren, 2011). While difficulties with the different sub-concepts have been demonstrated in several studies, little is known about the developmental trajectories of these sub-concepts.

Natural number magnitudes are easily and directly perceived by symbolic representations and can only be represented by one term. However, rational numbers do not to follow the same rules (DeWolf & Vosniadou, in this issue; Merenluoto & Lehtinen, 2004; Ni & Zhou, 2005; Schneider & Siegler, 2010). First, rational numbers can be represented by an infinite number of terms. Not only can the same magnitude be represented in both decimal and fraction form; there are an infinite number of ways to represent a magnitude within both of these forms (e.g. $0.5 = 0.50 = 0.500 = ½ = 2/4 = 3/6 = \ldots$). Prior to rational number learning, students can be sure that each term represents a single magnitude, and that each magnitude is represented by a single term. The expansion of numerical representation options represents a fundamental change in the number concept, one that can be difficult for students to traverse.

Second, the magnitudes of rational numbers are not immediately perceived, as is the case with natural numbers. Determining a fraction's magnitude requires the understanding that the magnitude is represented by a ratio between the two terms. Less coherent models of rational numbers may lead students to view fractions as two separate integers. This component based reasoning may lead to the over-use of natural number based reasoning (Merenluoto & Lehtinen, 2002, 2004). As well, decimal numbers do not follow the pattern that more digits indicate a larger magnitude (1.65 is smaller than 1.7), as is the case with natural numbers. Failures with the comparison of decimal numbers may also be due to students' component based comparisons of the terms before and after the decimal (as in $65 > 7$, therefore $1.65 > 1.7$) (see Durkin & Rittle-Johnson, in this issue). Because of these large differences in representation modes, students' prior concepts of natural numbers hinder the development of a mathematical understanding of rational number magnitudes. Instead, substantial conceptual change is needed.

Even more demanding conceptual change is necessary for students to understand the density of rational numbers (Hannula, Pehkonen, Maijala, & Soro, 2006; Merenluoto & Lehtinen, 2002; Vamvakoussi et al., 2011; Vamvakoussi & Vosniadou, 2004, 2010). The two main conceptual differences concerning concepts of density are a) the sequencing of numbers and b) the presence of a successor number. Learning the sequence of natural numbers, based on their understanding of successor terms, is one of the first numerical experiences most children have and students conceive of natural numbers as being discrete quantities (Ni & Zhou, 2005; Vosniadou, Vamvakoussi, & Skopeliti, 2008). However, rational numbers do not have a discrete sequence that can be reproduced. Relatedly, while with natural numbers it is always possible to distinguish the successor number of any number, this is untrue when the number concept is expanded to rational numbers. In total, these characteristics of rational and natural numbers lead to a large disparity in their conceptualization. It is possible to determine the terms between any two non-equal natural numbers. However, there are an infinite number of rational numbers in between any two non-equal rational numbers. Coming to understand rational numbers as dense sets without successors is a challenge involving substantial conceptual change in the nature of numbers (Hartnett & Gelman, 1998).

For both the comparison and density of rational numbers, previous research has outlined the concepts based on natural number that require fundamental conceptual change. However, little previous research has focused on the developmental stages that students go through in traversing this conceptual change. Adults and university students have been found to possess incorrect concepts about rational numbers indicating that many students do not ever successfully leave behind their natural number based concepts of rational numbers (DeWolf & Vosniadou, in this issue; Merenluoto & Lehtinen, 2004).

1.2. Latent Profile and Latent Transition Analyses

Latent Profile Analysis (LPA) and Latent Transition Analysis (LTA) are employed in present study in the hopes that they can provide new insight into the developmental trajectories related to rational number conceptual change. Latent variable mixture models provide two main advantages over traditional modeling, such as cluster analysis, for the modeling of a number of social and behavioral characteristics (Muthén, 2001; Nylund, Asparouhov, & Muthén, 2007). First, latent variable mixture models allow for the testing of the statistical fit of different classifications (e.g. 3 vs. 4 classes). Thus, latent variable mixture models could provide much needed added value over traditional cluster analysis in investigating conceptual change. If sufficient theoretical assumptions are used, prior to testing, to determine potential classification models, latent variable mixture modeling can corroborate the most statistically appropriate model with a theoretically sound classification system. In this way, latent variable mixture models may help confirm whether there are separate developmental stages, and what these stages look like, throughout the process of conceptual change with rational numbers.

Second, Latent Transition Analyses may be useful for detailing complex trajectories of conceptual change with rational numbers, as a result of the ability to determine what types of responses appear in different developmental stages. Latent variable mixture models have been shown to be useful for the classification of children's mental models of the earth (Straatemeier et al., 2008) and in analyzing the development of children's concepts of sinking and floating (Schneider & Hardy, 2013). LTA is able to assess the probability of subjects changing class membership across two or more time-points, which may be useful as a method to detail the multi-faceted nature of conceptual change (Lanza & Collins, 2008; Nylund, 2007). Thus, Latent Transition Analysis may help determine how students transition from early misconceptions of rational numbers to successful rational number conceptual change.

1.3. Research questions and hypotheses

We therefore report on a one-year longitudinal study of third- and fifth-grade students' conceptual knowledge of rational numbers. We administered a test of rational number knowledge at three time points, and use students' responses on these tests to create LPA and LTA models. We aim to answer the question of whether there can be found a distinct developmental pattern showing growth of conceptual knowledge of rational numbers that can explained by the conceptual change processes identified above. We expect students to display difficulty in reasoning about concepts of rational numbers, particularly concepts of density (Hypothesis 1a). We expect that it is possible to model different profiles of conceptual knowledge of magnitude representations and density of rational numbers that can be explained by theories of conceptual change with rational numbers (Hypothesis 1b). Developmentally, we expect limited growth of conceptual knowledge of rational numbers (Hypothesis 2a). Particular, we expect conceptual knowledge of density to be less well-developed than conceptual knowledge of magnitude representations (Hypothesis 2b).

2. Methods

2.1. Participants

In total, 263 students (141 female) from two primary schools in Southwest Finland participated in the study. Students were between the ages of 10–12 years (Grades 3–5) at the start of testing. Students were tested at the beginning and at the end of spring term