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In search for the natural number bias in secondary school students' interpretation of the effect of arithmetical operations

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ABSTRACT

Although rational numbers are an essential part of mathematical literacy, they cause many difficulties for students. A major cause is the natural number bias. We examined this natural number bias in secondary school students in two related studies. In Study 1, 8th graders judged the correctness of algebraic expressions that address the effect of operations. The higher accuracy level on congruent items than on incongruent items yielded clear evidence for the natural bias. However, this bias was only significant in multiplication and division items. Additional interview data showed that students doubted more about the applicability of natural number principles in items with addition and subtraction. In Study 2 we additionally confronted 10th and 12th graders with the same tasks. The results of the second study showed that the natural number bias unexpectedly did not decrease towards the end of secondary education and remained present in multiplication and division items.

1. Introduction

An essential part of mathematical literacy is a good understanding of rational numbers, and, more particularly, of fractions. Research has revealed a strong predictive relation between early knowledge of fractions and later mathematics achievement (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Siegler, Fazio, Bailey, & Zhou, 2012; Siegler, Thompson, & Schneider, 2011; see also Torbeyns, Schneider, Xin, & Siegler, 2014). Still, a large body of research shows that children and adults have a lot of difficulties dealing with various aspects of rational numbers (Cramer, Post, & delMas, 2002; Mazzocco & Devlin, 2008; Vamvakoussi, Van Dooren, & Verschaffel, 2012; Vamvakoussi & Vosniadou, 2004), a phenomenon that is often attributed to the "natural number bias" (Vamvakoussi et al., 2012).

We first discuss the theoretical and empirical background of this natural number bias and then we relate this phenomenon to two closely related research topics: the use of rules related to operations and the substitution of literal symbols.

1.1. The natural number bias

A major source of difficulty in learning rational numbers is the tendency to inappropriately apply natural number properties (Ni & Zhou, 2005). Some authors refer to this phenomenon as the whole number bias (e.g., Ni & Zhou, 2005). However, given that the focus of this article lays also on the *positive* character of the natural numbers – which may be ignored in situations where negative whole numbers are involved – we use the term natural number bias (Van Dooren, Van Hoof, Lijnen, & Verschaffel, 2012).

Already before formal instruction, children have formed an idea of what numbers are and how they behave. This idea is based on their daily informal experience with natural numbers (Vamvakoussi & Vosniadou, 2004). In the first years of primary school this natural-number based knowledge is formalized and systematized (Greer, 2004). As a result, once the mathematical concept of rational numbers is introduced in the classroom, problems and misconceptions occur when students encounter situations with rational numbers in which the principles and rules for natural numbers are no longer applicable (Gelman, 2000; Smith, Solomon, & Carey, 2005; Vamvakoussi & Vosniadou, 2004). This results in a tendency to make systematic errors when solving tasks where relying on the natural number knowledge leads to an incorrect answer – hereafter called *incongruent items* (Moss, 2005; Ni & Zhou, 2005; Smith et al., 2005; Vamvakoussi & Vosniadou, 2004). On the other hand, when the same students solve tasks with rational numbers where the answer obtained by correct

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reasoning is in line with the answer obtained by reasoning about natural numbers – hereafter called *congruent items* – the accuracy rates are much higher (Nunes & Bryant, 2008; Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013).

Based on an extensive search of the research literature, we found four main aspects on which natural numbers differ from rational numbers and that therefore may lead to systematic errors. The first aspect relates to density: While natural numbers are discrete (you can always point out which number comes next), rational numbers are dense (you cannot point out which number comes next, because between any two rational numbers are always infinitely many numbers). This leads to the common mistake that there are no (or finitely many) numbers between two pseudo-successive numbers (for example 1.2 and 1.3) (Merenluoto & Lehtinen, 2004; Vamvakoussi, Christou, Mertens, & Van Dooren, 2011).

The second aspect is the representation of rational numbers: While a natural number has a single symbolic representation, each rational number has an infinite number of possible symbolic representations. Research has shown that students often do not accept the possibility that a fraction and a decimal can represent the same number (Vamvakoussi et al., 2012) and moreover consider a fraction as two (natural) numbers instead of a single number. For example, students have difficulty considering the numbers 0.75 and 3/4 as the same number, because their representation is very different (e.g., Smith et al., 2005; Stafylidou & Vosniadou, 2004).

The third aspect is the way the number size can be determined. Research indicates that errors in size comparison tasks are frequently made because students wrongly assume that, just like natural numbers, "longer decimals are larger" and "shorter decimals are smaller" (Resnick et al., 1989). Because students have trouble seeing a fraction as one number instead of two separate numbers, they further tend to wrongly assume that a fraction's numerical value increases when its denominator, numerator, or both increase (Mamede, Nunes, & Bryant, 2005; Meert, Grégoire, & Noël, 2010, see also Vosniadou & DeWolf, 2014).

The fourth aspect, which is the focus of this article, concerns the effect of arithmetic operations on rational numbers. In the first years of elementary education, when students do arithmetic with natural numbers only, they construct the rules that multiplication and addition will always lead to a larger outcome while division and subtraction will always result in a smaller outcome. Typically, these rules are not stated explicitly in instruction, but students nonetheless deduce them from the multitude of experiences where this is indeed the case. The rules are moreover in line with students' primitive models of arithmetical operations (Fischbein, Deri, Nello, & Marino, 1985): The primitive model of addition is *putting together*, that of subtraction is *taking away*, multiplication is *repeated addition* and division is *equal sharing*. In the domain of rational numbers these primitive models and their accompanying rules do no longer necessarily apply. However, students may still rely on them, leading to mistakes such as accepting non-whole numbers as multiplicands but not as multipliers, or thinking that 0.99*5 will lead to an outcome larger than 5 (Vamvakoussi et al., 2012; Hasemann, 1981).

In this article we report on two studies that focused on this fourth aspect. Since we used in our studies items with algebraic symbols, we briefly review in the next section some findings from the research on algebra that are directly relevant for our own research as well.

1.2. Substitution of literal symbols in algebra

Research has shown that students have various difficulties interpreting literal symbols in algebra (Kieran, 2006). For instance, they face difficulties accepting that a literal symbol in this context takes its meaning in the domain of numbers – and is not, for example, merely an abbreviation of an object's name (e.g., "h" for "height") (e.g., Booth, 1984). When students then do start to associate literal symbols with numbers, they typically believe that a variable always stands for one single number, a specific "unknown" that is to be discovered (Asquith, Stephens, Knuth, & Alibali, 2007). Another difficulty occurs when these literal symbols have to be seen as standing for a rational number rather than for a natural number. Research shows that many secondary school students show a tendency to substitute these literal symbols in algebra only with natural numbers (Christou & Vosniadou, 2005, 2012; Van Dooren & Vamvakoussi, 2010). Christou and Vosniadou (2005), for instance, asked 8th and 9th graders to write down "numerical values that you think can be assigned to Q1:a, Q2:−b, Q3: 4d, Q4:1/d, Q5: a/b, Q6: a + a + a and Q7:k + 3" (Christou & Vosniadou, 2005, p. 454). They found that students strongly tend to interpret literal symbols as standing for natural numbers. Only one fourth gave the correct answer that the literal symbols could stand for all types of numbers and values, while about half of the students only gave natural numbers as substitutes.

1.3. The present research

In this article, we combined elements from the research on the effect of arithmetic operations on rational numbers and on substitution of literal symbols in algebra, with the aim to investigate, in two large-scale studies, if the natural number bias could also be found in secondary school students' interpretations of algebraic expressions that address the effect of operations. In a first study we investigated, first, if and to what extent 8th grade students – who are just introduced into expressions involving literal symbols – suffered from the natural number bias. Two additional questions were whether this effect was present to the same extent for all four operations and how students approached these algebraic expressions in order to come to a conclusion. To answer these three questions we collected both quantitative data (through a paper-and-pencil test) and qualitative data (through interviews) from the 8th grade students. In a second study we looked for an evolution with age in the natural number bias throughout secondary education. For that purpose, we compared the results of the 8th grade students from the first study with those of 10th and 12th grade students.

We are not the first to investigate the natural number bias in the domain of operations. However, in this investigation we extend the previous research in three ways. First, while the presence of primitive models of operations has amply been studied in elementary school children (e.g., Fischbein et al., 1985), and adults' understanding of the effect of operations has also been explored (e.g., Vamvakoussi, Van Dooren, & Verschaffel, 2013), this phenomenon has – to the best of our knowledge – not yet been investigated in the ages between childhood, where children are just taught rational numbers and still make a lot of mistakes (Ni & Zhou, 2005), and adulthood, wherein people rarely make mistakes but the natural number bias still reveals itself through reaction times (DeWolf & Vosniadou, 2011; Vamvakoussi et al., 2012). Second, a systematic comparison between different age groups in a single large-scale study has not been made yet. A comparative study with participants from different age groups allows to directly investigate whether the natural number bias disappears towards the end of secondary education. Third, as far as we know, no study has yet systematically compared the strength of the natural number bias for the four different operations.

2. Study 1

Two kinds of data were collected. First, a collective paper-and-pencil test was taken from a large group of 8th graders ($N = 291$),