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Commentary

The development of rational number knowledge: Old topic, new insights

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ABSTRACT

The development of rational number knowledge has been studied extensively by mathematics education researchers, cognitive-developmental psychologists and, more recently, by neuroscientists as well. Building on a rich body of prior research, and some exciting new ideas, the target articles re-visit several topics, with a view to refine and deepen our understanding of how rational number understanding develops. The effect of prior natural number knowledge-either positive or adverse-on rational number learning is highlighted by all contributors. I draw on the articles to discuss five different aspects of the whole, or natural, number bias.

1. Introduction

This special issue addresses the development of rational number knowledge,[1] an issue that has been studied extensively by mathematics education researchers, cognitive-developmental psychologists and, more recently, has attracted the interest of neuroscientists as well (e.g., Jacob, Vallentin, & Nieder, 2012). Building on a rich body of prior research, and some exciting new ideas, the contributors re-visit several topics, with a view to refine and deepen our understanding of how rational number understanding is developed. Kelley and Rittle-Johnson study misconceptions about decimal numbers in connection to the individual's confidence about the response; McMullen, Laakkonen, Hannula-Sormunen, and Lehtinen study the development of students' understanding of density in a longitudinal study and with the use of new statistical techniques; Van Hoof, Vandewalle, Verschaffel, and Van Dooren take a closer look at students' interpretation of literal symbols and their understanding of the effect of operations combined; DeWolf and Vosniadou, and Torbeyns, Schneider, and Siegler look into fraction magnitude representations with a view to support two different, albeit not incompatible, theoretical views.

2. Rational numbers are difficult … but why?

Let me start this commentary with a point stressed in all contributions, namely that rational numbers are challenging for students. Drawing on empirical evidence as well as conceptual analyses coming for numerous studies, Moss (2005) summarizes several reasons why rational numbers are difficult: Students need to construct a complex knowledge network based on multiplicative rather than on additive relations; new symbols and representations are introduced that need to be understood and coordinated; the notion of the unit and of the arithmetical operations need to be reconceptualised; and there are several conceptually distinct meanings attached to rational numbers that, again, need to be understood and coordinated. These include the part-whole aspect of fraction, fraction as a quotient, fraction as a multiplicative operator, fraction as a ratio, and fraction as measure. The latter is closely related to an aspect that is particularly relevant to this special issue: Rational numbers are *numbers*, that is, abstract entities that take their meaning within a number system, through their relations with other numbers and in accordance with certain principles and rules, such as the basic laws of arithmetic (e.g., Kilpatrick, Swafford, & Findell, 2001). There is a huge difference between abstract and concrete conceptualizations of number (both historically and developmentally). Indeed, “three apples are more than two apples” is not the same as “3 is bigger than 2”; similarly, “half of an apple is more than one quarter of the apple” is not the same as “1/2 is bigger than 1/4”. The latter, abstract, conceptualization is far more challenging (Kilpatrick et al., 2001; Ni & Zhou, 2005). All five articles of this special issue address precisely this

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[1] For the purposes of this commentary, the term rational number is used to refer to numbers that are in, or can be converted to, the form a/b, where a and b are integers and b is non-zero. This is consistent with the school mathematics definition of the rational numbers, and by no means equivalent to the formal definition of the rational numbers.

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abstract conceptualization of number: Participants are asked to compare decimals (Kelley & Rittle-Johnson) and fractions (DeWolf & Vosniadou, McMullen et al., 2015; Torbeyns et al., 2015); and place fractions on the number line (Torbeyns et al., 2015). It is only in this abstract context that numbers can be understood as densely ordered (McMullen et al., 2015). Furthermore, this kind of abstraction is required in order to eventually conceptualize rational numbers as a unified number system (Kilpatrick et al., 2001), which is necessary in order to assign more than one type of numbers to real variables and judge the effect of operations with unknown numbers (Van Hoof et al., 2015).

Moss's (2005) summary illustrates nicely two main points that are relevant to the articles of this special issue: Rational numbers are difficult for students, because a lot of new material has to be learnt, and the content is highly complex (even without considering the vast variety of related applications). Furthermore, prior knowledge and experience with natural numbers is not always supportive of rational number learning, a phenomenon noticed, studied, and reported by many mathematics education researchers (see the studies cited in Kilpatrick et al., 2001; Moss, 2005; Ni & Zhou, 2005; Vamvakoussi, Vosniadou, & Van Dooren, 2013), far before the term "whole number bias" was coined by Ni and Zhou. Thus, Torbeyns, Schneider, Xin, and Siegler (2015) are right in arguing that natural number knowledge interference is one, but not the only source of difficulty in rational number learning – and let me add that the brief summary above indicates that the picture is even more complicated than depicted in their article. There is no doubt, however, that natural number knowledge interference is one major source of conceptual difficulties.

3. Theoretical framing of the contributions

The idea of a whole or natural number bias (hereafter, natural number bias) is closely related to the problem of restructuring a prior knowledge base that cannot adequately support a new, and in many ways incompatible, number perspective (Ni & Zhou, 2005; Vamvakoussi, Van Dooren, & Verschaffel, 2012); it is thus related to conceptual change perspectives on the development of rational number knowledge. With the exception of Torbeyns et al. (2015), all contributions are framed in conceptual change terms, focusing on the differences between natural and rational numbers, and studying the complex interactions between students' prior knowledge and the information about rational numbers coming from instruction. Since my view on the matter is expressed in detail elsewhere (e.g., Vamvakoussi & Vosniadou, 2010), and since conceptual change perspectives are also addressed by the contributors, I will focus on the theoretical position of Torbeyns et al. who advocate a different, albeit not incompatible idea. Specifically, Torbeyns et al. focus on the similarities, rather than the differences, between natural and rational numbers. Adopting the integrated theory of numerical development (Siegler, Thompson, & Schneider, 2011), they in fact address the question "what makes natural and non-natural numbers members of the same category, the category of number?" Their answer is "magnitude". Thus numerical development is described as a process of progressively broadening the class of numbers that are understood to possess magnitudes, are subject to ordering, and can be assigned specific locations on number lines. Siegler et al. made two assumptions: a) fractions are crucial for overall mathematical understanding, and b) understanding magnitudes is crucial for understanding fractions, which is also tested-and supported-by Torbeyns et al. in the present cross-cultural study.

Let me start by saying that I find the idea of exploiting the deep similarities between natural and rational numbers valuable, particularly in terms of educational implications (Vamvakoussi et al., 2013). In fact, this idea has been systematically explored by several mathematics education researchers (e.g., Behr, Harel, Post, & Lesh, 1994; Sophian, 2004; Steffe & Olive, 2010). It is also hard to disagree that magnitude is an essential part of the meaning of number in an abstract context. Yet there is another, possibly more fundamental, commonality, between natural and rational numbers, namely the notion of the unit. The use of the unit differentiates between judgments regarding unquantified and quantified quantities and is thus instrumental for the development of number concepts (Sophian, 2008). The ability to choose or construct appropriate units is considered fundamental for the development of rational number concepts (Harel & Confrey, 1994), notably for the understanding of fraction as measure. And although this fact is often overlooked, the notion of the unit is also essential for natural numbers as well (e.g., Sophian, 2004).

Number magnitude depends crucially on the unit. I believe that this fact is reflected in the findings of Torbeyns et al. (2015). Indeed, one might ask, why is fraction magnitude estimation on the number line (particularly on the 0–5 number line) more difficult than fraction comparisons, and a better predictor of overall mathematical achievement? I would argue that this is because it requires understanding of the role of the unit. This is essential in concrete contexts, such as measurement (Nunes & Bryant, 1996); and more so in abstract ones, such as placing a fraction on the number line. I would not be surprised if several students in the sample of Torbeyns et al. treated the length corresponding to 5 as the unit (see also Ni, 2000).

This said, it is valuable to have research-based evidence that fraction understanding correlates with overall mathematical achievement. One possible explanation of this finding is the pervasiveness of rational number ideas in the curriculum. Rational numbers are an important part of what Vergnaud (1994) termed "multiplicative conceptual field" that includes notions ranging from basic ones such as multiplication and division, to highly sophisticated ones such as *n*-linear functions. The elements of the multiplicative conceptual field are interrelated, and there is wide variety of mathematical concepts that relate to this field within as well as outside school settings. For instance, proportionality, geometrical similarity, and probability all pertain to the multiplicative conceptual field (see Lamon, 2006, for a detailed discussion). Acknowledging the interconnections between rational number ideas and a wide variety of mathematical notions, many of which are taught at school, one can expect that fraction understanding is important for students' mathematical achievement in different countries (Torbeyns et al., 2015) – and arguably for students' achievement in physics and chemistry, as well.

I would agree with Torbeyns et al. (2015) that conceptual change perspectives focus on a particular (and indeed limited) aspect of the development of rational number knowledge. However, such perspectives (notably, the framework developed by Vosniadou and colleagues, and presented by DeWolf & Vosniadou, 2015) provide detailed accounts of the basis of the natural number bias; predictions of what happens when prior knowledge interacts with new information-and these predictions become more refined (e.g., Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015), as there is now a substantial body of prior research; descriptions and explanations of students' conceptions and how these change (or do not change) with instruction (e.g., Kelley & Rittle-Johnson, 2015; McMullen, Laakkonen, Hannula-Sormunen, & Lehtinen, 2015). In fact, intense research from this perspective has put forward several aspects of the natural number bias, which I will discuss in the following section.

The integrated theory, on the other hand, has a more ambitious goal, is a more recent attempt, and thus has more questions to address, in particular *how* do students come to conceptualize