Artificial Intelligence 174 (2010) 1007–1026

Contents lists available at ScienceDirect

Artificial Intelligence

www.elsevier.com/locate/artint

On the measure of conflicts: Shapley Inconsistency Values ☆

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A R T I C L E   I N F O

Article history:
Received 16 July 2007
Received in revised form 9 June 2010
Accepted 9 June 2010
Available online 18 June 2010

Keywords:
Inconsistency management
Inconsistency tolerance
Inconsistency measures
Conflict resolution
Paraconsistency
Shapley values

A B S T R A C T

There are relatively few proposals for inconsistency measures for propositional belief bases. However inconsistency measures are potentially as important as information measures for artificial intelligence, and more generally for computer science. In particular, they can be useful to define various operators for belief revision, belief merging, and negotiation. The measures that have been proposed so far can be split into two classes. The first class of measures takes into account the number of formulae required to produce an inconsistency: the more formulae required to produce an inconsistency, the less inconsistent the base. The second class takes into account the proportion of the language that is affected by the inconsistency: the more propositional variables affected, the more inconsistent the base. Both approaches are sensible, but there is no proposal for combining them. We address this need in this paper: our proposal takes into account both the number of variables affected by the inconsistency and the distribution of the inconsistency among the formulae of the base. Our idea is to use existing inconsistency measures in order to define a game in coalitional form, and then to use the Shapley value to obtain an inconsistency measure that indicates the responsibility/contribution of each formula to the overall inconsistency in the base. This allows us to provide a more reliable image of the belief base and of the inconsistency in it.

1. Introduction

There are numerous works on reasoning under inconsistency. One can quote for example paraconsistent logics, argumentation frameworks, belief revision and fusion, etc. All these approaches illustrate the fact that the dichotomy between consistent and inconsistent sets of formulae that comes from classical logics is not sufficient for describing these sets. As shown by these works, normally when given two inconsistent sets of formulae, they are not trivially equivalent. They do not contain the same information and they do not contain the same contradictions.

Measures of information *à la* Shannon have been studied in logical frameworks (see for example [31]). Roughly they involve counting the number of models of the set of formulae (the less models, the more informative the set). The problem is that these measures regard an inconsistent set of formulae as having a null information content, which is counter-intuitive (especially given all the proposals for paraconsistent reasoning). So generalizations of measures of information have been proposed to solve this problem [39,53,36,32,24].

In comparison, there are relatively few proposals for inconsistency measures [22,27,35,32,28,18]. However, these measures are potentially important in diverse applications in artificial intelligence, such as belief revision, belief merging, and negotiation, and more generally in computer science. Already some provisional studies indicate that measuring inconsistency

☆ This paper is a revised and extended version of the paper "Shapley Inconsistency Values" presented at KR'06.

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0004-3702/$ – see front matter
doi:10.1016/j.artint.2010.06.001

may be seen to be a useful tool in analysing a diverse range of information types including news reports [29], integrity constraints [18], software specifications [9,10,42], and ecommerce protocols [12].

The current proposals for measuring inconsistency can be classified in two approaches. The first approach involves "counting" the minimal number of formulae needed to produce the inconsistency. The more formulae needed to produce the inconsistency, the less inconsistent the set [35]. This idea is an interesting one, but it rejects the possibility of a more fine-grained inspection of the (content of the) formulae. In particular, if one looks to singleton sets only, one is back to the initial problem, with only two values: consistent or inconsistent.

The second approach involves looking at the proportion of the language that is touched by the inconsistency. This allows us to look *inside* the formulae [27,32,18]. This means that two formulae viewed as two whole belief bases (singleton sets) can have different inconsistency measures. But, in these approaches one can identify the set of formulae with its conjunction (i.e. the set $\{\varphi, \varphi'\}$ has the same inconsistency measure as the set $\{\varphi \wedge \varphi'\}$). This can be sensible in some applications, but this means that the distribution of the contradiction among the formulae is not taken into account.

What we propose in this paper is a definition for inconsistency measures that allow us to take the best of the two approaches. This will allow us to build inconsistency measures that are able to look inside the formulae, but also to take into account the distribution of the contradiction among the different formulae of the set.

The above-mentioned approaches define *inconsistency measures*, i.e. functions that associate a number to each belief base. These global base-level measures are sufficient for a variety of applications. But in some cases we need an evaluation on a finer level, that is for each formula of the base. We call these functions, that associate a number to each formula of a base, *inconsistency values*. Such a function allows us to identify which are the most problematic formulae of a belief base with respect to the inconsistency. This can be very useful for applications such as belief revision or negotiation. These inconsistency values provide a more detailed view of the inconsistency, and they can be used to defined new inconsistency measures which more accurately reflect the inconsistency of the whole base.

To this end we will use a notion that comes from coalitional game theory: the Shapley value. This value assigns to each player the payoff that this player can expect from her utility for each possible coalition. The idea is to use existing inconsistency measures (that allow us to look inside the formulae) in order to define a game in coalitional form, and then to use the Shapley value to obtain an inconsistency measure with the desired properties. From these inconsistency values, it is possible to define new interesting inconsistency measures. We present these measures, we state a set of logical properties they satisfy, and we show that they are more interesting than the other existing measures.

The plan of the paper is as follows: After some preliminaries in the next section, Section 3 introduces inconsistency measures that count the number of formulae needed to produce an inconsistency. Section 4 presents the approaches where the inconsistency measure depends on the number of variables touched by the inconsistency. Section 5 introduces the problem studied in this paper and illustrates that the naive solution is not adequate. Section 6 gives the definition of coalitional games and of the Shapley value. Section 7 introduces the inconsistency measures based on Shapley value. Then we study the logical properties of these measures in Section 8, and we provide a complete axiomatization of a particular measure in Section 9 through a set of intuitive axioms. Section 10 sketches the possible applications of those measures for reasoning and for belief change operators. Finally Section 11 concludes by giving perspectives of this work and its possible applications for belief change operators.

2. Preliminaries

We will consider a propositional language $\mathcal{L}$ built from a finite set of propositional symbols $\mathcal{P}$. We will use $a, b, c, \ldots$ to denote the propositional variables, and Greek letters $\alpha, \beta, \varphi, \ldots$ to denote the formulae. An interpretation is a total function from $\mathcal{P}$ to $\{0, 1\}$. The set of all interpretations is denoted $\mathcal{W}$. An interpretation ω is a model of a formula φ, denoted $\omega \models \varphi$, if and only if it makes φ true in the usual truth-functional way. $Mod(\varphi)$ denotes the set of models of the formula φ, i.e. $Mod(\varphi) = \{\omega \in \mathcal{W} \mid \omega \models \varphi\}$. We will use $\subseteq$ to denote the set inclusion, and we will use $\subset$ to denote the strict set inclusion, i.e. $A \subset B$ iff $A \subseteq B$ and $B \nsubseteq A$. Let A and B be two subsets of C, we note $C = A \oplus B$ if A and B form a partition of C, i.e. $C = A \oplus B$ iff $C = A \cup B$ and $A \cap B = \emptyset$. We will denote the set of real numbers by $\mathbb{R}$.

A *belief base* K is a finite set of propositional formulae. More exactly, as we will need to identify the different formulae of a belief base in order to associate them with their inconsistency value, we will consider belief bases K as vectors of formulae. For logical properties we will need to use the set corresponding to each vector, so we suppose that there is a function mapping each vector $K = (\alpha_1, \ldots, \alpha_n)$ into $\bar{K}$, the set $\{\alpha_1, \ldots, \alpha_n\}$. As it will never be ambigous, in the following we will omit the graphical distinction and write K as both the vector and the set.

Let us note $\mathcal{K}_\mathcal{L}$ the set of belief bases definable from formulae of the language $\mathcal{L}$. A belief base is consistent if there is at least one interpretation that satisfies all its formulae.

If a belief base K is not consistent, then one can define the minimal inconsistent subsets of K as:

$$\mathsf{MI}(K) = \{K' \subseteq K \mid K' \vdash \bot \text{ and } \forall K'' \subset K', K'' \nvdash \bot\}$$

If one wants to recover consistency from an inconsistent base K by removing some formulae, then the minimal inconsistent subsets can be considered as the purest form of inconsistency. To recover consistency, one has to remove at least one formula from each minimal inconsistent subset [49]. The notion of maximal consistent subset is the dual of that of