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A minimal contractor for the polar equation: Application to robot localization

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ABSTRACT

Contractor programming relies on a catalog on elementary contractors which need to be as efficient as possible. In this paper, we introduce a new theorem that can be used to build minimal contractors consistent with equations, and another new theorem to derive an optimal separator from a minimal contractor. As an application, we focus on the channeling polar constraint associated to the change between Cartesian coordinates and Polar coordinates. We illustrate our method on the localization problem of an actual underwater robot where both range and goniometric measurements of landmarks are collected.

1. Introduction

Contractor programming (Chabert and Jaulin, 2009) is an efficient tool to solve rigorously complex problems involving uncertainties and nonlinear equations (Ceberio and Granvilliers., 2001; Kreinovich et al., 1997). A *contractor* $\mathcal{C}_{\mathbb{X}}$ is an operator able to contract a box of $\mathbb{R}^n$ without removing a single point of the subset $\mathbb{X}$ of $\mathbb{R}^n$ to which it is associated. As a result, using a paving of $\mathbb{R}^n$ generated by a paver (Sainudiin, 2010), the contractor will allow us to build an outer approximation of $\mathbb{S}$. Basic notions on interval analysis, contractors and applications can be found in Jaulin et al. (2001).

Contractor programming relies on a catalog of elementary contractors. Most of the time, these elementary contractors are built using interval arithmetic (Moore, 1966). Then, by combining all these elementary contractors, we can construct a more sophisticated contractor consistent with the solution set of the problem we want to solve. The principle can be extended to separator programming (Jaulin and Desrochers, 2014) in order to compute an inner and an outer approximation of the solution set.

Now, combining contractors introduces a pessimism which has to be balanced by additional bisections performed by the paver. For more efficiency, it is important to extend the catalog by adding some new specific contractors.

In this paper, we propose some new theorems in order to build more easily optimal contractors/separators consistent with equations often used, for instance, in the field of robotics (Kieffer et al., 1999; Daney et al., 2006; Langerwisch and Wagner, 2012). As an application, we will consider the *polar* constraint associated to the change of coordinates between Cartesian and polar form (Candau et al., 2006). This polar constraint is essential for localization of robots when both goniometric and distance measurements are available (Colle, 2013; Di Marco et al., 2001). Some test cases will show that our approach makes it possible to obtain an inner and an outer approximation of the solution set in a much more efficient manner than simply composing elementary interval contractors.

This paper is organized as follows. Section 2 presents the notion of contractor and separator algebra. Section 3 shows how a minimal contractor for some specific constraints can be built. Section 4 will then derive an optimal separator for the polar transformation. Section 5 provides an application on the localization of an actual underwater robot and Section 6 concludes the paper.

2. Contractors and separators

This section recalls the basic notions on intervals, contractors

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Table 1
Notations.

Subsets of $\mathbb{R}^n$	$\mathbb{X}, \mathbb{Y}$
Intervals of $\mathbb{R}$	$[a]$
Boxes of $\mathbb{R}^n$	$[\mathbf{a}]$
Set of boxes of $\mathbb{R}^n$	$\mathbb{IR}^n$
Box hull of a set $\mathbb{A}$	$[\![\mathbb{A}]\!]$
Union hull of two boxes	$[\mathbf{x}]\sqcup[\mathbf{y}] = [\![[\mathbf{x}] \cup [\mathbf{y}]]\!]$
Composition of functions	$\mathbf{f}\circ\mathbf{f} = \mathbf{f}^2$

and separators that are needed to understand the contribution of this paper. An *interval* of $\mathbb{R}$ is a closed connected set of $\mathbb{R}$. A box $[x]$ of $\mathbb{R}^n$ is the Cartesian product of n intervals. The set of all boxes of $\mathbb{R}^n$ is denoted by $\mathbb{IR}^n$. Notations used in this paper are given in Table 1.

2.1. Contractors and separators

In this section, we recall the basic notions on contractors and separators that will be used later. A *contractor* $\mathcal{C}$ is an operator $\mathbb{IR}^n \mapsto \mathbb{IR}^n$ (see e.g., Di-Loreto et al., 2007) such that

$$\begin{array}{ll} \mathcal{C}([x]) \subset [x] & \text{(contractance)} \\ [x] \subset [y] \Rightarrow \mathcal{C}([x]) \subset \mathcal{C}([y]). & \text{(monotonicity)} \end{array} \quad (1)$$

We define the inclusion between two contractors $\mathcal{C}_1$ and $\mathcal{C}_2$ as follows:

$$\mathcal{C}_1 \subset \mathcal{C}_2 \Leftrightarrow \forall [x] \in \mathbb{IR}^n, \quad \mathcal{C}_1([x]) \subset \mathcal{C}_2([x]). \quad (2)$$

A set $\mathbb{X}$ is *consistent* (see Fig. 1) with the contractor $\mathcal{C}$ (we will write $\mathbb{X} \sim \mathcal{C}$) if for all $[x]$, we have

$$\mathcal{C}([x]) \cap \mathbb{X} = [x] \cap \mathbb{X}. \quad (3)$$

Two contractors $\mathcal{C}$ and $\mathcal{C}_1$ are equivalent (we will write $\mathcal{C} \sim \mathcal{C}_1$) if we have:

$$\mathbb{X} \sim \mathcal{C} \Leftrightarrow \mathbb{X} \sim \mathcal{C}_1. \quad (4)$$

A contractor $\mathcal{C}$ is *minimal* if for any other contractor $\mathcal{C}_1$, we have the following implication:

$$\mathcal{C} \sim \mathcal{C}_1 \Rightarrow \mathcal{C} \subset \mathcal{C}_1. \quad (5)$$

If $\mathcal{C}$ is a minimal contractor consistent with $\mathbb{X}$, then for all $[x]$, we have $\mathcal{C}([x]) \cap \mathbb{X} = [\![[x] \cap \mathbb{X}]\!]$ where $[\![\mathbb{A}]\!]$ is the *hull* operator, i.e., the smallest box which encloses $\mathbb{A}$. This means that $\mathcal{C}([x])$ corresponds exactly to the smallest box that can be obtained by a contraction of $[x]$ without removing a single point of $\mathbb{X}$. As a consequence, there exists a unique minimal contractor.

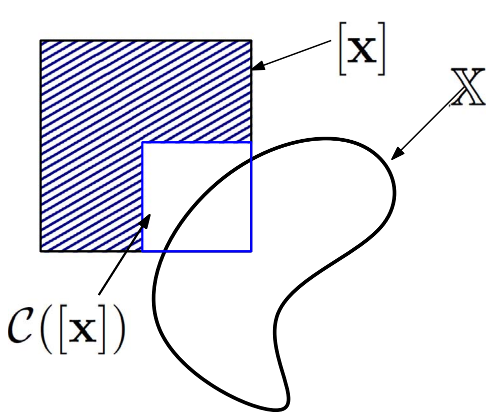

Fig. 1. Contractor consistent with to the set $\mathbb{X}$.

Example 1. The minimal contractor $\mathcal{C}_{\mathbb{X}}$ consistent with the set

$$\mathbb{X} = \left\{ x \in \mathbb{R}^2, (x_1 - 2)^2 + (x_2 - 2.5)^2 \in [1, 4] \right\} \quad (6)$$

can be built using a forward-backward constraint propagation (Benhamou et al., 1999; Drevelle and Bonnifait, 2012). The contractor $\mathcal{C}_{\mathbb{X}}$ can be used by a paver to obtain an outer approximation for $\mathbb{X}$. This is illustrated in Fig. 2 (left) where $\mathcal{C}_{\mathbb{X}}$ removes parts of the space outside $\mathbb{X}$ (painted light-gray). But due to the consistency property (see Eq. (3)) $\mathcal{C}_{\mathbb{X}}$ has no effect on boxes included in $\mathbb{X}$. A box partially included in $\mathbb{X}$ cannot be eliminated and is bisected, except if its length is larger than an given value ε. The contractor $\mathcal{C}_{\mathbb{X}}$ only provides an outer approximation of $\mathbb{X}$.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two contractors, we define the following operations on contractors (Chabert and Jaulin, 2009):

$$(\mathcal{C}_1 \cap \mathcal{C}_2)([x]) = \mathcal{C}_1([x]) \cap \mathcal{C}_2([x]) \quad (7)$$

$$(\mathcal{C}_1 \sqcup \mathcal{C}_2)([x]) = \mathcal{C}_1([x]) \sqcup \mathcal{C}_2([x]) \quad (8)$$

$$(\mathcal{C}_1 o \mathcal{C}_2)([x]) = \mathcal{C}_1(\mathcal{C}_2([x])) \quad (9)$$

where $\sqcup$ is the *union hull* defined by

$$[x] \sqcup [y] = [\![[x] \cup [y]]\!]. \quad (10)$$

In order to characterize an inner and outer approximation of the solution set, we introduce the notion of *separator*.

A *separator* $\mathcal{S}$ is a pair of contractors $\{\mathcal{S}^{\text{in}}, \mathcal{S}^{\text{out}}\}$ such that, for all $[x] \in \mathbb{IR}^n$, we have

$$\mathcal{S}^{\text{in}}([x]) \cup \mathcal{S}^{\text{out}}([x]) = [x] \quad \text{(complementarity)}. \quad (11)$$

A set $\mathbb{X}$ is *consistent* with the separator $\mathcal{S}$ (we will write $\mathbb{X} \sim \mathcal{S}$), if

$$\mathbb{X} \sim \mathcal{S}^{\text{out}} \text{ and } \overline{\mathbb{X}} \sim \mathcal{S}^{\text{in}}, \quad (12)$$

where $\overline{\mathbb{X}} = \{x \,\|\, x \notin \mathbb{X}\}$. This notion of separator is illustrated in Fig. 3. We define the inclusion between two separators $\mathcal{S}_1$ and $\mathcal{S}_2$ as follows:

$$\mathcal{S}_1 \subset \mathcal{S}_2 \Leftrightarrow \mathcal{S}_1^{\text{in}} \subset \mathcal{S}_2^{\text{in}} \text{ and } \mathcal{S}_1^{\text{out}} \subset \mathcal{S}_2^{\text{out}}. \quad (13)$$

A separator $\mathcal{S}$ is *minimal* if

$$\mathcal{S}_1 \subset \mathcal{S} \Rightarrow \mathcal{S}_1 = \mathcal{S}. \quad (14)$$

It is trivial to check that $\mathcal{S}$ is minimal implies that the two contractors $\mathcal{S}^{\text{in}}$ and $\mathcal{S}^{\text{out}}$ are both minimal. If we define the following operations:

$$\begin{array}{lcll} \mathcal{S}_1 \cap \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\text{in}} \cup \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cap \mathcal{S}_2^{\text{out}}\right\} & \text{(intersection)} \\ \mathcal{S}_1 \cup \mathcal{S}_2 & = & \left\{\mathcal{S}_1^{\text{in}} \cap \mathcal{S}_2^{\text{in}}, \mathcal{S}_1^{\text{out}} \cup \mathcal{S}_2^{\text{out}}\right\} & \text{(union)} \end{array} \quad (15)$$

then we have (Jaulin and Desrochers, 2014)

$$\left\{\begin{array}{lcl} \mathcal{S}_1 & \sim & \mathbb{X}_1 \\ \mathcal{S}_2 & \sim & \mathbb{X}_2 \end{array}\right. \Rightarrow \left\{\begin{array}{lcl} \mathcal{S}_1 \cap \mathcal{S}_2 & \sim & \mathbb{X}_1 \cap \mathbb{X}_2 \\ \mathcal{S}_1 \cup \mathcal{S}_2 & \sim & \mathbb{X}_1 \cup \mathbb{X}_2 \end{array}\right. \quad (16)$$

Example 2. Consider the set $\mathbb{X}$ of Example 1. From the contractor consistent with