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Approximate solving of nonlinear ordinary differential equations using least square weight function and metaheuristic algorithms

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ABSTRACT

Differential equations play a noticeable role in engineering, physics, economics, and other disciplines. In this paper, a general approach is suggested to solve a wide variety of linear and nonlinear ordinary differential equations (ODEs) that are independent of their forms, orders, and given conditions. With the aid of certain fundamental concepts of mathematics, Fourier series expansion and metaheuristic methods, ODEs can be represented as an optimization problem. The target is to minimize the weighted residual function (cost function) of the ODEs. To this end, two different approaches, unit weight function and least square weight function, are examined in order to determine the appropriate method. The boundary and initial values of ODEs are considered as constraints for the optimization model. Generational distance metric is used for evaluation and assessment of the approximate solutions versus the exact solutions. Six ODEs and four mechanical problems are approximately solved and compared with their exact solutions. The optimization task is carried out using different optimizers including the particle swarm optimization, the cuckoo search, and the water cycle algorithm. The optimization results obtained show that metaheuristic algorithms can be successfully applied for approximate solving of different types of ODEs. The suggested least square weight function is slightly superior over the unit weight function in terms of accuracy and statistical results for approximate solving of ODEs.

1. Introduction

Mathematical formulation of most physical and engineering problems involves differential equations (DEs). DEs have applications in all areas of science and engineering. Hence, it is important for engineers and scientists to know how to set up DEs and solve them. DEs are categorized into two types: ordinary differential equations (ODEs) and partial differential equations (PDEs).

An ODE is a DE in which all the derivatives are with respect to a single independent variable, whereas in PDEs, the derivatives are with respect to multiple variables (Boyce and Diprima, 1997). In this paper, we modeled ODEs as optimization problems and determined their approximate solutions.

With regard to real life problems, which are highly nonlinear, many problems in engineering and science often include one or more ODEs. Indeed, behavior of any system predicted by a model can be represented by ODEs.

Analytical approaches are often inefficient in tackling ODEs (except the simple ODEs). Therefore, approximate analytical methods are applied to obtain solutions of ODEs. A number of analytical methods has been utilized to develop approximate analytical solutions for engineering problems, such as the variational iteration method (VIM) (Coşkun and Atay, 2007, 2008), the homotopy analysis method (Domairry and Fazeli, 2009), the method of bilaterally bounded (MBB) (Lee, 2006), and the Adomian double decomposition method (Chiu and Chen, 2002; Arslanturk, 2009).

Differential transformation method (DTM), which is based on Taylor series expansion, was first introduced by Zhou (1986) for solving linear and nonlinear initial value problems in electrical circuits. The DTM has been widely used to obtain approximate solutions for nonlinear engineering problems (Rashidi et al., 2010; Kundu and Barman, 2010; Yaghoobi and Torabi, 2011; Torabi et al., 2012).

Recently, many studies have combined the concept of the DTM with finite difference approximation for increasing the capability of their approximate solution (Chu and Chen, 2008; Chu and Lo, 2008; Peng and Chen, 2011). Further, approximate analytical procedures such as the VIM, the homotopy perturbation method, and DTM have been coupled with the Padé approximation technique to overcome the disadvantages faced by these methods in certain cases (Torabi et al., 2013).

In particular, many studies have applied approximation methods for solving various types of integro-differential equations (linear/nonlinear) (Yalcinbas and Sezer, 2000; Darania and Abadian, 2006; Darania and Ebadian, 2007; Darania and Ivaz, 2008; Roul and Meyer,

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2011). However, each of these numerical approximation techniques has its own operational limitations that strictly narrow its functioning domain. Hence, it is strongly possible that these approximate techniques fail to overcome a specific problem.

A few such instances are mentioned in the following (e.g., for heat transfer problems). It was reported that the DTM was unable to produce physically reasonable data for the Glauert-jet problem (Torabi et al., 2012). Moreover, for some specific parameter values, the HPM and VIM failed to provide accurate results for the motion of a solid particle in a fluid (Torabi and Yaghoobi, 2011; Yaghoobi and Torabi, 2012). Meanwhile, these approximation methods are based on classical mathematical tools.

Metaheuristic algorithms are usually devised by observing the phenomena occurring in nature. With the emergence of metaheuristic algorithms, complex problems are not distant from finding their solutions. Metaheuristic optimization algorithms have demonstrated their capabilities in finding near-optimal solutions to numerical real-valued problems, for which exact and analytical methods may not be able to produce better solutions within reasonable computation time (Osman and Laporte, 1996; Glover and Kochenberger, 2003; Yang, 2010a, 2010b).

Nowadays, applications that use metaheuristic methods for finding approximate solution of ODEs have increased considerably. This includes genetic algorithms (GAs) (Mateescu, 2006; Mastorakis, 2007), particle swarm optimization (PSO) (Lee, 2006; Babaei, 2013), genetic programming (Cao et al., 2000), and others (Reich, 2000; Karr and Wilson, 2003).

Despite there being a wide range of approximate methods for solving ODEs, there is a lack of a general approach that meets all the engineering demands having unconventional, nonlinear ODEs. It should be very interesting to solve linear and nonlinear ODEs having arbitrary boundaries and/or initial values using a single approach.

Recently, the concept of Fourier series expansion has been used as a base approximate function for finding the approximate solution of ODEs. Hence, the ODEs problem was modeled as an optimization problem and solved using the PSO (Babaei, 2013). For simplicity, the weight function considered in the literature was set to unit weight function (Babaei, 2013). However, this assumption may not help us in obtaining better results for all types of ODEs.

In this paper, using the concept of Fourier series as the base approximate function, other optimizers including the PSO (Kennedy and Eberhart, 1995), the cuckoo search (CS) (Yang and Deb, 2009), and the water cycle algorithm (WCA) (Eskandar et al., 2012) are applied for optimization purposes. Also, least square weight function is proposed for solving ODEs. The obtained results are compared with the unit weight function in terms of statistical results and performance metric for 10 ODEs.

The remainder of this paper is organized as follows: The next section describes the approximate approach for tackling ODEs and the suggested weighted residual function. In addition, the performance criterion for quantitative assessment among other methods is given in Section 2. In Section 3, short descriptions of the applied optimizers are given. Section 4 describes 10 ODE test problems along with their best results, and graphical comparisons between the exact and approximate solutions. Statistical optimization results of three applied optimizers for different weight functions are given in Section 5. In this section, the reported test problems have been compared in terms of performance metric and different weight functions. Finally, conclusions are drawn in Section 6.

2. Approximate method for ODEs

In this section, a general approach for solving ODEs is given detail. In mathematics, an ODE is an equality involving a function and its derivatives. An ODE of order n is an equation having the following form:

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{1}$$

where y is a function of x, $y' = dy/dx$ is the first derivative with respect to x, and $y^{(n)} = d^{(n)}y/dx^{(n)}$ is the nth derivative with respect to x. Nonhomogeneous ODEs can be solved if the general solution to the homogenous version is known (Boyce and Diprima, 1997).

2.1. Approximate base function

Based on the suggestion in the literature (Babaei, 2013), Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sine and cosine terms. Fourier series employs the orthogonal relationships of the sine and cosine functions.

The computation and study of Fourier series is known as harmonic analysis. This is extremely useful to decompose an arbitrary periodic function into a set of simple terms. Using some basic concepts of mathematics, accompanied with Fourier expansion, and metaheuristic methods, it is straightforward to solve different types of ODEs having different nature (linear/nonlinear).

In the traditional method for solving DEs, most of the methods are invented to handle the first or second order ODEs, initial and boundary value problems. However, using the proposed approximate method, there are no such limitations. In fact, using this approximate method in combination with metaheuristic optimizers, there is always an acceptable solution for any type of ODEs for higher orders in their implicit forms.

Generally, using Fourier series, one can estimate any periodic function having finite terms. In this paper, we used Fourier series as our base approximate function. There are three main reasons for this selection which are as follows: (1) A powerful theory backs its convergence for a wide variety of continuous functions (Kreyszig, 2009), (2) It contains sine and cosine terms, which are differentiable up to any order, and (3) A unique form of the approximation function can be utilized for different kinds of ODEs.

To clarify further, consider the implicit form of a nonlinear ODE given in Eq. (1), which has to be solved for the interval span x_0 to x_n, having boundary and initial conditions as defined in Eqs. (2) and (3), respectively:

$$y(x_0) = y_0, \quad y'(x_0) = y'_0 \ldots \quad y(x_n) = y_n, \quad y'(x_n) = y'_n \ldots, \tag{2}$$

$$y(x_0) = y_0, \quad y'(x_0) = y'_0 \ldots \quad y^{(n-1)}(x_0) = y_0^{(n-1)}. \tag{3}$$

In general, the suggested approximate base function, which is the partial sum of the Fourier series (finite simple terms of sine and cosine functions) with center x_0, as follows (Kreyszig, 2009):

$$y(x) \approx Y_{appx}(x) = a_0 + \sum_{m=1}^{NT} \left[a_m \cos\left(\frac{m\pi(x-x_0)}{L}\right) + b_m \sin\left(\frac{m\pi(x-x_0)}{L}\right) \right]. \tag{4}$$

Accordingly, the derivatives of Eq. (4) are given as follows:

$$y'(x) \approx Y'_{appx}(x) = \sum_{m=1}^{NT} \left[-\frac{m\pi}{L} a_m \sin\left(\frac{m\pi(x-x_0)}{L}\right) + \frac{m\pi}{L} b_m \cos\left(\frac{m\pi(x-x_0)}{L}\right) \right]$$
$$y''(x) \approx Y''_{appx}(x) = \sum_{m=1}^{NT} \left[-\left(\frac{m\pi}{L}\right)^2 a_m \cos\left(\frac{m\pi(x-x_0)}{L}\right) - \left(\frac{m\pi}{L}\right)^2 b_m \sin\left(\frac{m\pi(x-x_0)}{L}\right) \right]$$
$$\vdots$$
$$y^{(n)}(x) \approx Y^{(n)}_{appx}(x) = \sum_{m=1}^{NT} \left[\left(\frac{m\pi}{L}\right)^n a_m \cos\left(\frac{m\pi(x-x_0)}{L} + \frac{n\pi}{2}\right) + \left(\frac{m\pi}{L}\right)^n b_m \sin\left(\frac{m\pi(x-x_0)}{L} + \frac{n\pi}{2}\right) \right]. \tag{5}$$

In order to create a weighted residual function ($R(x)$), Eqs. (4) and (5) are replaced in Eq. (1). In these equations (Eq. (1)–(5)), x_0 and x_n are the lower and upper bounds of the interval solution; L is the