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Brief paper

Introduction to the algebra of separators with application to path planning

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ABSTRACT

Contractor algebra is a numerical tool based on interval analysis which makes it possible to solve many nonlinear problems arising in robotics, such as identification, path planning or robust control. This paper presents a new notion of *separators* which is a pair of complementary contractors and presents the corresponding algebra. Using separator algebra inside a paver will allow us to get an inner and an outer approximation of the solution set in a much simpler way than using any other interval approach. A path planning problem will then be considered in order to illustrate the principle of the approach.

1. Introduction

Many problems in engineering amount to characterizing a set of $\mathbb{R}^n$ defined by constraints (see Veres and Wall, 2000 for a general introduction to set-membership approaches). For instance, the solution set may correspond (i) to the set of all parameters that are consistent with some interval measurements (Lévêque et al., 1997; Gning and Bonnifait, 2006; Kreinovich et al., 1997; Gning et al., 2013; Chabert and Jaulin, 2009a), (ii) to the set of all configuration vectors such that a robot does not meet any obstacle (Porta et al., 2007; Jaulin, 2001), (iii) to the set of all parameter vectors of a controller such that the closed loop system is stable (Wan et al., 2009; Didrit et al., 1995), (iv) to the set of all calibration parameters (Daney et al., 2006; Ramdani and Poignet, 2005), (v) to attractors of dynamical systems (Tucker, 1999). More formally, the problem to be considered in this paper is to bracket a set $\mathbb{X}$ defined by constraints between two sets $\mathbb{X}^-$ and $\mathbb{X}^+$ such that

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+. \quad (1)$$

The set $\mathbb{X}$ is assumed to be defined as combinations of atomic sets. These atomic sets may correspond to sets defined by any nonlinear inequalities or geometric sets such as a map. To compute such an approximation of $\mathbb{X}$, two main classes of approaches are considered: symbolic and numerical. The symbolic approach provides a set of methods based on computer algebra that are guaranteed, efficient, but limited to polynomial problems. The numerical approaches are mainly composed with linear methods (for which linear algebra can be used, see, *e.g.*, Rokityanski and Veres, 2005), convex methods (such as those based on semidefinite programming Henrion et al., 2009), Monte-Carlo methods (which can be used for a large class of nonlinear problems Thrun et al., 2005) and interval methods (Moore, 1966; Jaulin et al., 2001) which provide algorithms to compute inner and outer subpavings (*i.e.*, union of non-overlapping boxes) to approximate $\mathbb{X}$. The principle of interval methods is similar to Monte Carlo except that they compute with boxes (thanks to interval arithmetic Moore, 1966) instead of points, so they can guarantee that all the search space has been covered. Interval methods can thus deal with a large class of nonlinear problems (larger than for semidefinite or symbolic methods) in a guaranteed way (contrary to Monte-Carlo). The main drawbacks of interval methods are the high complexity with respect to the number of unknown variables and the lack of tools/software that make it possible to implement efficiently an interval resolution dedicated to a nonlinear problem. In order to allow a resolution of high dimensional problems and to facilitate the implementation of efficient interval methods, the notion of a contractor (Chabert and Jaulin, 2009b) has recently been introduced. A *contractor* is an operator which is able to contract boxes without removing any solution. Contractor-based techniques (Chabert and Jaulin, 2009b) combined with a paver (*i.e.*, a bisection algorithm which partitions the research space with boxes) can provide an outer subpaving approximation $\mathbb{X}^+$ of $\mathbb{X}$ in an efficient

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way. For the inner subpaving $\mathbb{X}^-$, the *De Morgan* rules can be used to express the complementary set $\overline{\mathbb{X}}$ of $\mathbb{X}$. Then basic contractor techniques can be used to get an inner characterization $\mathbb{X}^-$. Now, the task is not so easy and has never been made automatic. The contribution of this paper is to introduce a new mathematical object, named *separator*, which is composed of two complementary contractors: an inner contractor and an outer contractor. These two contractors are computed together by focusing on the boundary of $\mathbb{X}$. An algebra, similar to the algebra developed for contractors (Chabert and Jaulin, 2009b), will then be developed for these separators. This will make it possible to easily build separators associated with complex sets $\mathbb{X}$. Combined with a paver, separators will then be able to bracket $\mathbb{X}$ between the two subpavings $\mathbb{X}^-$ and $\mathbb{X}^+$ in an easy way and without asking the programmer to build both the inner and the outer contractors.

The paper is organized as follows. Section 2 defines separators and shows how they can be used inside a paver to characterize subsets of $\mathbb{R}^n$. Section 3 explains how to extend all basic operations on sets (such as union, intersection, difference, complementary) to separators. The inversion of separators through a vector function is treated in Section 4. Section 5 presents how atomic separators can be defined. More complex separators will be obtained by compositions of these atomic separators. An application related to path planning is considered in Section 6. Section 7 concludes the paper.

2. Separators

In this section, we first present the notion of contractors (Jaulin et al., 2001) that will be needed to define separators. Then, we show how separators can be used by a paver in order to bracket the solution set $\mathbb{X}$ between two subpavings $\mathbb{X}^-$ and $\mathbb{X}^+$.

2.1. Contractors

An *interval* of $\mathbb{R}$ is a closed connected set of $\mathbb{R}$. A box $[\mathbf{x}]$ of $\mathbb{R}^n$ is the Cartesian product of n intervals. The set of all boxes of $\mathbb{R}^n$ is denoted by $\mathbb{IR}^n$. A *contractor* $\mathcal{C}$ is an operator $\mathbb{IR}^n \mapsto \mathbb{IR}^n$ such that

$$\begin{array}{ll} \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance)} \\ [\mathbf{x}] \subset [\mathbf{y}] \;\Rightarrow\; \mathcal{C}([\mathbf{x}]) \subset \mathcal{C}([\mathbf{y}]) & \text{(monotonicity)}. \end{array} \quad (2)$$

We define the inclusion between two contractors $\mathcal{C}_1$ and $\mathcal{C}_2$ as follows:

$$\mathcal{C}_1 \subset \mathcal{C}_2 \Leftrightarrow \forall [\mathbf{x}] \in \mathbb{IR}^n, \quad \mathcal{C}_1([\mathbf{x}]) \subset \mathcal{C}_2([\mathbf{x}]). \quad (3)$$

A set $\mathbb{S}$ is *consistent* with the contractor $\mathcal{C}$ (we will write $\mathbb{S} \sim \mathcal{C}$) if for all $[\mathbf{x}]$, we have

$$\mathcal{C}([\mathbf{x}]) \cap \mathbb{S} = [\mathbf{x}] \cap \mathbb{S}. \quad (4)$$

Two contractors $\mathcal{C}$ and $\mathcal{C}_1$ are *consistent* with each other (we will write $\mathcal{C} \sim \mathcal{C}_1$) if for any set $\mathbb{S}$, we have

$$\mathbb{S} \sim \mathcal{C} \Leftrightarrow \mathbb{S} \sim \mathcal{C}_1. \quad (5)$$

A contractor $\mathcal{C}$ is *minimal* if for any other contractor $\mathcal{C}_1$, we have the following implication:

$$\mathcal{C} \sim \mathcal{C}_1 \Rightarrow \mathcal{C} \subset \mathcal{C}_1. \quad (6)$$

We define the *negation* $\neg\mathcal{C}$ of a contractor $\mathcal{C}$ as follows:

$$\neg\mathcal{C}([\mathbf{x}]) = \{\mathbf{x} \in [\mathbf{x}] \mid \mathbf{x} \notin \mathcal{C}([\mathbf{x}])\}. \quad (7)$$

Note that $\neg\mathcal{C}([\mathbf{x}])$ is not a box in general, but a union of boxes.

2.2. Separators

A *separator* $\mathcal{S}$ is pair of contractors $\{\mathcal{S}^{in}, \mathcal{S}^{out}\}$ such that, for all $[\mathbf{x}] \in \mathbb{IR}^n$, we have

$$\mathcal{S}^{in}([\mathbf{x}]) \cup \mathcal{S}^{out}([\mathbf{x}]) = [\mathbf{x}] \quad \text{(complementarity)}. \quad (8)$$

A set $\mathbb{S}$ is *consistent* with the separator $\mathcal{S}$ (we will write $\mathbb{S} \sim \mathcal{S}$), if

$$\mathbb{S} \sim \mathcal{S}^{out} \quad \text{and} \quad \overline{\mathbb{S}} \sim \mathcal{S}^{in}. \quad (9)$$

where $\overline{\mathbb{S}} = \{\mathbf{x} \mid \mathbf{x} \notin \mathbb{S}\}$. We define the *remainder* of a separator $\mathcal{S}$ as

$$\partial\mathcal{S}([\mathbf{x}]) = \mathcal{S}^{in}([\mathbf{x}]) \cap \mathcal{S}^{out}([\mathbf{x}]). \quad (10)$$

Note that the remainder is a contractor and not a separator.

Example. Fig. 1 represents a set $\mathbb{S}$, an outer contractor $\mathcal{S}^{out}$, an inner contractor $\mathcal{S}^{in}$ and a boundary contractor $\partial\mathcal{S}$. The pair $\{\mathcal{S}^{in}, \mathcal{S}^{out}\}$ corresponds to a separator associated with $\mathbb{S}$ (painted gray on the figure). Note that $\mathcal{S}^{in}$ is only allowed to eliminate the part of the research space which is inside $\mathbb{S}$ whereas $\mathcal{S}^{out}$ only

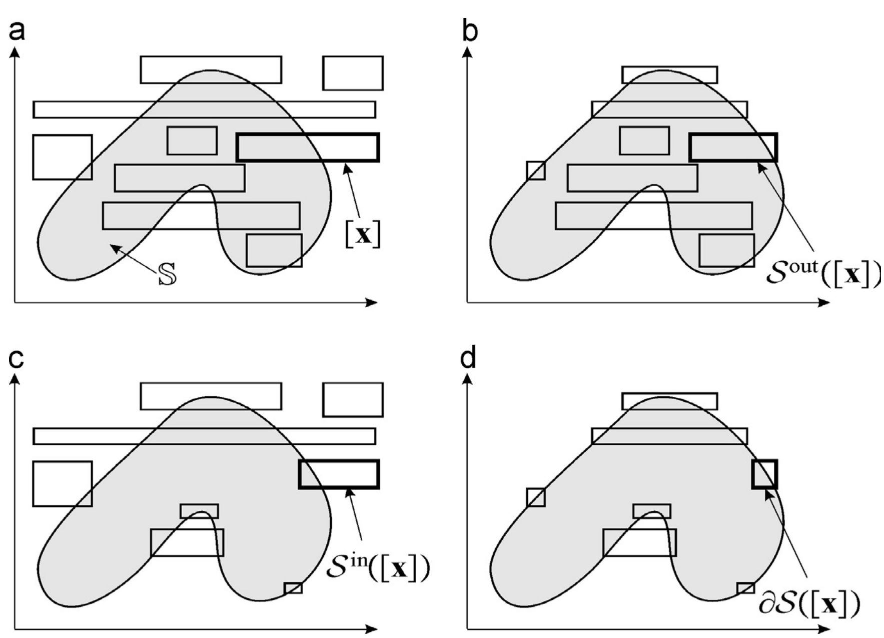

Fig. 1. A separator $\{\mathcal{S}^{in}, \mathcal{S}^{out}\}$ is a pair of two contractors.