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Resolution of nonlinear interval problems using symbolic interval arithmetic

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A B S T R A C T

An interval problem is a problem where the unknown variables take interval values. Such a problem can be defined by interval constraints, such as "the interval $[a,b] \subset [a,b]^2$". Interval problems often appear when we want to analyze the behavior of an interval solver. To solve interval problems, we propose to transform the constraints on intervals into constraints on their bounds. For instance, the previous interval constraint $[a,b] \subset [a,b]^2$ can be transformed into the following bound constraints "$a \geq \min(a^2, ab, b^2)$ and $b \leq \max(a^2, ab, b^2)$". Classical interval solvers can then be used to solve the resulting bound constraints. The procedure which transforms interval constraints into equivalent bound constraints can be facilitated by using symbolic interval arithmetic. While classical intervals can be defined as a pair of two real numbers, symbolic intervals can be defined as a pair of two symbolic expressions. An arithmetic similar to classical interval arithmetic can be defined for symbolic intervals. The approach will be illustrated on several applications.

1. Introduction

Interval analysis (Moore, 1979) is an efficient numerical tool to solve nonlinear problems such as global optimization (Hansen, 1992), set characterization (Jaulin et al., 2001), etc. in a reliable way. Although interval methods made it possible to solve efficiently a large class of nonlinear *punctual problems* (i.e., problems where the solutions to be found are vectors or real numbers), they also brought new questions and new problems about the properties and the behaviors of the interval algorithms. Most of these new problems can be cast into the framework of *interval problems*, i.e., problems where the solution set is composed with intervals or boxes.

This paper introduces symbolic intervals with its arithmetic. The idea is similar to that of numerical interval computation: the interval operations are replaced by operations on their bounds. But for symbolic intervals, these operations are performed in a symbolic way. This symbolic arithmetic will make possible to transform an interval problem into a punctual problem in a systematic way. The resulting punctual problem will then be solved using classical numerical interval methods.

To our knowledge, the idea of applying interval arithmetic rules in a symbolic way has never been proposed before. Of course, in the context of interval methods, classic symbolic calculus was already used to improve the efficiency of interval solvers (see, e.g., van Emden, 1999), but the interval rules was only applied on the numerical resolution, not on the symbolic part.

Section 2 shows how an interval problem can be transformed into an equivalent punctual problem. Symbolic interval arithmetic is introduced in Section 3. Section 4 presents some potential applications of symbolic intervals. Most of them cannot be solved with existing tools, to our knowledge. Section 5 concludes the paper.

2. Interval problem and bound problem

2.1. Interval constraint

An interval $[x]$ is a closed bounded set of $\mathbb{R}$. The set of all intervals is denoted by $\mathbb{IR}$. A box $[\mathbf{x}] = [x_1] \times \cdots \times [x_1]$ of $\mathbb{R}^n$ is the Cartesian product of n intervals. The set of all boxes of $\mathbb{R}^n$ is denoted by $\mathbb{IR}^n$. An interval constraint is a function from $\mathbb{IR}^n$ to $\{0,1\}$, where 0 and 1 stand for false and true, respectively. An example of interval constraint is

$$C([\mathbf{x}]) : [x_1] \subset [x_2], \tag{1}$$

where $[\mathbf{x}] = [x_1] \times [x_2]$. An interval constraint is *monotonic* if

$$[\mathbf{x}] \subset [\mathbf{y}] \Rightarrow (C([\mathbf{x}]) \Rightarrow C([\mathbf{y}])).$$

For instance $C([x]) \overset{\text{def}}{=} t(0 \in [x])$ is monotonic.

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2.2. Intervalization function

Define the *intervalization* function $\imath$ as follows:

$$\imath : \begin{cases} \mathbb{R}^{2n} \to \mathbb{IR}^n \\ \underbrace{\begin{pmatrix} x_1^- \\ x_1^+ \\ \vdots \\ x_n^- \\ x_n^+ \end{pmatrix}}_{\underline{\overline{\mathbf{x}}}} \to \underbrace{\begin{pmatrix} [x_1^-, x_1^+] \\ \vdots \\ [x_n^-, x_n^+] \end{pmatrix}}_{[\mathbf{x}] = \imath(\underline{\overline{\mathbf{x}}})} \quad \text{if } \forall j, x_j^- \le x_j^+. \end{cases}$$

The domain of $\imath$ is

$$\mathrm{dom}(\imath) = \{\underline{\overline{\mathbf{x}}} = (x_1^-, x_1^+, \ldots, x_n^-, x_n^+), \forall j, x_j^- \le x_j^+\} = \imath^{-1}(\mathbb{IR}^n).$$

Note that the bijection $\imath$ between $\mathrm{dom}(\imath)$ and $\mathbb{IR}^n$ is a classical and useful conceptual tool (see Kulpa, 2006) to develop original interval algorithms. To an interval constraint $C([\mathbf{x}])$ from $\mathbb{IR}^n$ to $\{0,1\}$ we can define the corresponding bound constraint $\overline{\underline{C}}(\underline{\overline{\mathbf{x}}}) = C \circ \imath(\underline{\overline{\mathbf{x}}})$ on the bound vector $\underline{\overline{\mathbf{x}}} \in \mathbb{R}^{2n}$:

$$\overline{\underline{C}} : \begin{cases} \mathbb{R}^{2n} \to \mathbb{IR}^n \to \{0,1\} \\ \underbrace{\begin{pmatrix} x_1^- \\ x_1^+ \\ \vdots \\ x_n^- \\ x_n^+ \end{pmatrix}}_{\underline{\overline{\mathbf{x}}}} \overset{\imath}{\to} \underbrace{\begin{pmatrix} [x_1^-, x_1^+] \\ \vdots \\ [x_n^-, x_n^+] \end{pmatrix}}_{[\mathbf{x}]} \overset{C}{\to} C([\mathbf{x}]). \end{cases}$$

For instance, to the interval constraint

$$C([\mathbf{x}]) \overset{\text{def}}{=} ([x_1] \subset [x_2]),$$

we associate the following bound constraint:

$$\overline{\underline{C}} \begin{pmatrix} x_1^- \\ x_1^+ \\ x_2^- \\ x_2^+ \end{pmatrix} : \begin{cases} x_1^- \ge x_2^- & \text{and} \\ x_1^+ \le x_2^+ & \text{and} \\ x_1^- \le x_1^+ & \text{and} \\ x_2^- \le x_2^+. \end{cases}$$

Symbolic intervals, to be presented in the following section, is a new symbolic tool that makes possible to transform interval problems into a problem on the bounds of the intervals.

3. Symbolic intervals

3.1. Definition

A *symbolic interval* is a pair, denoted by $[\mathcal{A}^-, \mathcal{A}^+]$, of two mathematical expression $\mathcal{A}^-$ and $\mathcal{A}^+$. For instance

$$[\sin(a+b), a^2+b]$$

is a symbolic interval. Its lower bound is the expression $\mathcal{A}^- = \sin(a+b)$ and its upper bound is the expression $\mathcal{A}^+ = a^2+b$.

3.2. Operations

We define the operations on symbolic intervals as classical interval operations, but in a symbolic way.

$$[\mathcal{A}^-, \mathcal{A}^+] + [\mathcal{B}^-, \mathcal{B}^+] = [\mathcal{A}^- + \mathcal{B}^-, \mathcal{A}^+ + \mathcal{B}^+],$$

$$[\mathcal{A}^-, \mathcal{A}^+] - [\mathcal{B}^-, \mathcal{B}^+] = [\mathcal{A}^- - \mathcal{B}^+, \mathcal{A}^+ - \mathcal{B}^-],$$

$$[\mathcal{A}^-, \mathcal{A}^+] * [\mathcal{B}^-, \mathcal{B}^+] = [\min(\mathcal{A}^- * \mathcal{B}^-, \mathcal{A}^- * \mathcal{B}^+, \mathcal{A}^+ * \mathcal{B}^-, \mathcal{A}^+ * \mathcal{B}^+), \max(\mathcal{A}^- * \mathcal{B}^-, \ldots)],$$

$$[\mathcal{A}^-, \mathcal{A}^+]^2 = [\max(0, \mathrm{sign}(\mathcal{A}^- * \mathcal{A}^+))\min((\mathcal{A}^-)^2, (\mathcal{A}^+)^2), \max((\mathcal{A}^-)^2, (\mathcal{A}^+)^2)],$$

$$\exp([\mathcal{A}^-, \mathcal{A}^+]) = [\exp(\mathcal{A}^-), \exp(\mathcal{A}^+)],$$

$$1/[\mathcal{A}^-, \mathcal{A}^+] = [\min(1/\mathcal{A}^+, \infty * (\mathcal{A}^- * \mathcal{A}^+)), \max(1/\mathcal{A}^-, -\infty * (\mathcal{A}^- * \mathcal{A}^+))],$$

$$[\mathcal{A}^-, \mathcal{A}^+] \cap [\mathcal{B}^-, \mathcal{B}^+] = [\max(\mathcal{A}^-, \mathcal{B}^-), \min((\mathcal{A}^+, \mathcal{B}^+))],$$

$$[\mathcal{A}^-, \mathcal{A}^+] \sqcup [\mathcal{B}^-, \mathcal{B}^+] = [\min(\mathcal{A}^-, \mathcal{B}^-), \max((\mathcal{A}^+, \mathcal{B}^+))],$$

$$w([\mathcal{A}^-, \mathcal{A}^+]) = \mathcal{A}^+ - \mathcal{A}^-.$$

For instance,

$$\begin{aligned} &\exp([a+b, a^2+b] - [\sin(ab), a+b]) \\ &\quad = \exp([a+b-(a+b), a^2+b-\sin(ab)]) \\ &\quad = [\exp(a+b-(a+b)), \exp(a^2+b-\sin(ab))]. \end{aligned}$$

Note that some operations on symbolic intervals have not the same form as their classical numerical counterpart. They have been rewritten in order to get a symbolic interval as the result of the operation, i.e., a pair of two symbolic expressions. For instance, the definition of the square of an interval $[a,b]$ is classically defined by

$$\begin{aligned} [a,b]^2 &= [0, \max(a^2, b^2)] \quad \text{if } 0 \in [a,b] \\ &= [\min(a^2, b^2), \max(a^2, b^2)] \quad \text{if } 0 \notin [a,b]. \end{aligned}$$

Now, since

$$\begin{aligned} \max(0, \mathrm{sign}(ab)) &= 0 \quad \text{if } 0 \in [a,b] \\ &= 1 \quad \text{if } 0 \notin [a,b], \end{aligned}$$

we can define $[a,b]^2$ by a single expression

$$[a,b]^2 = [\max(0, \mathrm{sign}(ab))\min(a^2, b^2), \max(a^2, b^2)].$$

This explains the definition of $[\mathcal{A}^-, \mathcal{A}^+]^2$ for symbolic intervals.

3.3. Relations

We also extend classical interval relations to symbolic intervals:

$$([\mathcal{A}^-, \mathcal{A}^+] = [\mathcal{B}^-, \mathcal{B}^+]) \Leftrightarrow \mathcal{A}^- - \mathcal{B}^- = 0, \quad \mathcal{A}^+ - \mathcal{B}^+ = 0$$

$$([\mathcal{A}^-, \mathcal{A}^+] \subset [\mathcal{B}^-, \mathcal{B}^+]) \Leftrightarrow \mathcal{A}^- - \mathcal{B}^- \ge 0, \quad \mathcal{B}^+ - \mathcal{A}^+ \ge 0.$$

For instance

$$\begin{aligned} &([a+b, a^2+b] = [\sin(ab), a-b]) \\ &\Leftrightarrow (a+b = \sin(ab) \text{ and } a^2+b = a-b). \end{aligned}$$

Another example is the following:

$$([a,b] \subset [a,b]^2) \Leftrightarrow \begin{cases} a - \max(0, \mathrm{sign}(a \cdot b)) * \min(a^2, b^2) \ge 0, \\ \max(a^2, b^2) - b \ge 0. \end{cases}$$

3.4. Simplification

The expression involved as lower and upper bound of symbolic intervals are classical expressions over real variables. Thus classical simplification techniques can be applied to symbolic interval. For instance, we can write

$$[a-b+a+b, a^2+c-a*a] = [2a, c]. \tag{2}$$

But note that

$$[a,b] - [a,b] = [a-b, b-a] \tag{3}$$

which is not [0,0]. The simplifications can only be performed on the expressions inside the interval and not on the symbolic interval operations.